Elliptic surfaces and B-M obstructions

Harvey Yau

March 19, 2024

Section 1

Elliptic surfaces

Elliptic surfaces

We usually express an elliptic curve with a Weierstrass equation

$$E : \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Elliptic surfaces

We usually express an elliptic curve with a Weierstrass equation

$$E : \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

If we want to study families of elliptic curves, we may want to allow the coefficients to vary as functions in a variable t, i.e. to have $a_i \in k(t)$.

Elliptic surfaces

We usually express an elliptic curve with a Weierstrass equation

$$E: \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

If we want to study families of elliptic curves, we may want to allow the coefficients to vary as functions in a variable t, i.e. to have $a_i \in k(t)$. Then, for almost every value of t, we can evaluate each a_i at t to obtain an elliptic curve

$$E_t: \quad y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t).$$

(Note that we also need the discriminant to be nonzero.)

Elliptic surfaces

We usually express an elliptic curve with a Weierstrass equation

$$E: \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

If we want to study families of elliptic curves, we may want to allow the coefficients to vary as functions in a variable t, i.e. to have $a_i \in k(t)$. Then, for almost every value of t, we can evaluate each a_i at t to obtain an elliptic curve

$$E_t: \quad y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t).$$

(Note that we also need the discriminant to be nonzero.) We can also view E as an elliptic curve defined over the field $k(t)$.

Elliptic surfaces

To encompass all of these, we can view the equation as defining an *elliptic surface* $\mathcal{E}$, whose points are triples (x, y, t) satisfying the equation given earlier.

Elliptic surfaces

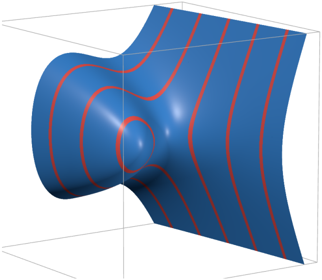
To encompass all of these, we can view the equation as defining an *elliptic surface* $\mathcal{E}$, whose points are triples (x, y, t) satisfying the equation given earlier.

This surface $\mathcal{E}$ also comes with a *fibration*, which is a morphism $\mathcal{E} \to \mathbb{P}^1_t$ sending the point (x, y, t) to the point on $\mathbb{P}^1$ with coordinate t.

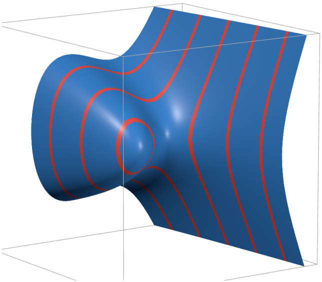
Elliptic surfaces

To encompass all of these, we can view the equation as defining an *elliptic surface* $\mathcal{E}$, whose points are triples (x, y, t) satisfying the equation given earlier.

This surface $\mathcal{E}$ also comes with a *fibration*, which is a morphism $\mathcal{E} \to \mathbb{P}^1_t$ sending the point (x, y, t) to the point on $\mathbb{P}^1$ with coordinate t.

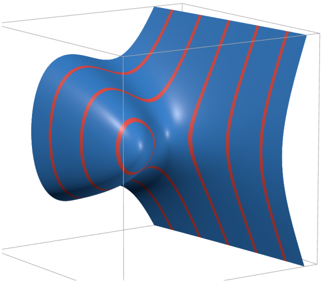


Elliptic surfaces



- The curve E_t can be found inside the surface $\mathcal{E}$ as the fibre $\pi^{-1}(t)$ above the point t.

Elliptic surfaces



- The curve E_t can be found inside the surface $\mathcal{E}$ as the fibre $\pi^{-1}(t)$ above the point t.
- The curve E is the generic fibre $\pi^{-1}(\eta)$ of the fibration.

Elliptic surfaces and the generic fibre

The (scheme-theoretic) points of the generic fibre E are also points of the elliptic surface $\mathcal{E}$.

Elliptic surfaces and the generic fibre

The (scheme-theoretic) points of the generic fibre E are also points of the elliptic surface $\mathcal{E}$.

- Generic point of $E \leftrightarrow$ Generic point of $\mathcal{E}$

Elliptic surfaces and the generic fibre

The (scheme-theoretic) points of the generic fibre E are also points of the elliptic surface $\mathcal{E}$.

- Generic point of $E \leftrightarrow$ Generic point of $\mathcal{E}$
- Rational point of $E \leftrightarrow$ Section of $\mathcal{E}$

Elliptic surfaces and the generic fibre

The (scheme-theoretic) points of the generic fibre E are also points of the elliptic surface $\mathcal{E}$.

- Generic point of $E \leftrightarrow$ Generic point of $\mathcal{E}$
- Rational point of $E \leftrightarrow$ Section of $\mathcal{E}$
- Closed point of $E \leftrightarrow$ Irreducible horizontal curve of $\mathcal{E}$

Elliptic surfaces and the generic fibre

The (scheme-theoretic) points of the generic fibre E are also points of the elliptic surface $\mathcal{E}$.

- Generic point of $E \leftrightarrow$ Generic point of $\mathcal{E}$
- Rational point of $E \leftrightarrow$ Section of $\mathcal{E}$
- Closed point of $E \leftrightarrow$ Irreducible horizontal curve of $\mathcal{E}$

An example

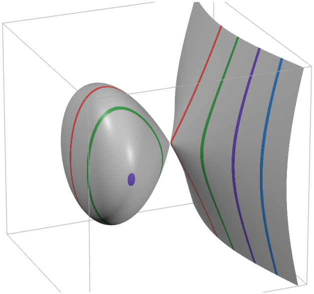
Consider the surface given by the equation

$$y^2 = x^3 + 3x^2 - t^2.$$

An example

Consider the surface given by the equation

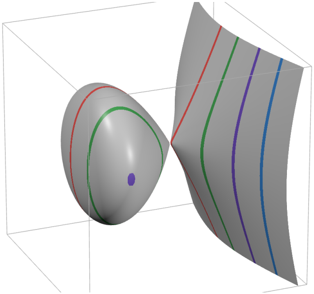
$$y^2 = x^3 + 3x^2 - t^2.$$



An example

Consider the surface given by the equation

$$y^2 = x^3 + 3x^2 - t^2.$$



We see for example that the fibre above $t = 1$ is an elliptic curve, given by equation $y^2 = x^3 + 3x^2 - 1$. We call this a *good fibre*.

An example - horizontal curves

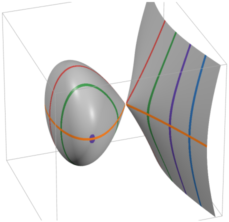
The 2-torsion of the curve E is given by the three roots of the cubic $x^3 + 3x^2 - t^2 = 0$. However, this cubic is irreducible over $k(t)$.

An example - horizontal curves

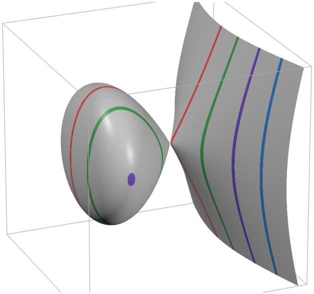
The 2-torsion of the curve E is given by the three roots of the cubic $x^3 + 3x^2 - t^2 = 0$. However, this cubic is irreducible over $k(t)$. So the three 2-torsion points are Galois conjugates of each other, and hence correspond to the same scheme-theoretic point of E.

An example - horizontal curves

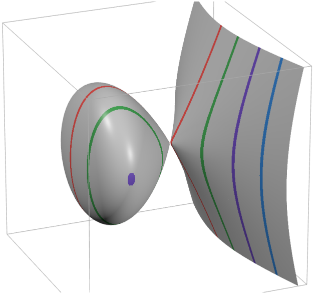
The 2-torsion of the curve E is given by the three roots of the cubic $x^3 + 3x^2 - t^2 = 0$. However, this cubic is irreducible over $k(t)$. So the three 2-torsion points are Galois conjugates of each other, and hence correspond to the same scheme-theoretic point of E.
The corresponding horizontal curve of $\mathcal{E}$ will be the curve that passes through the 2-torsion points of each fibre.



An example - bad fibres

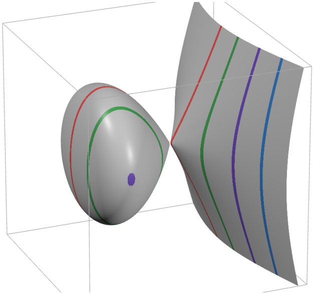


An example - bad fibres



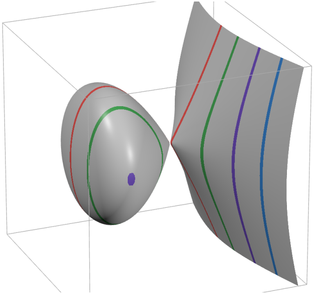
The fibre above $t = 2$ is not an elliptic curve, it is a nodal cubic. We call this a *bad fibre*.

An example - bad fibres



The fibre above $t = 2$ is not an elliptic curve, it is a nodal cubic. We call this a *bad fibre*. Note that even though the node is a singular point of the fibre E_2, it is a nonsingular point of $\mathcal{E}$.

An example - bad fibres



The fibre above $t = 2$ is not an elliptic curve, it is a nodal cubic. We call this a *bad fibre*. Note that even though the node is a singular point of the fibre E_2, it is a nonsingular point of $\mathcal{E}$.
The fibre above $t = 0$ is also a nodal cubic. This time, the node is a singular point of $\mathcal{E}$.

Resolution of singularities

For our purposes, we will want to study elliptic surfaces that are smooth (i.e. nonsingular). However, most of the time, the surface $\mathcal{E}$ we obtain from writing down an equation in Weierstrass form will not be smooth.

Resolution of singularities

For our purposes, we will want to study elliptic surfaces that are smooth (i.e. nonsingular). However, most of the time, the surface $\mathcal{E}$ we obtain from writing down an equation in Weierstrass form will not be smooth. To remedy this, we will resolve the singularity by *blowing it up*. This will give us a new surface that looks the same as $\mathcal{E}$ everywhere except for the singularity, and is smooth.

Blowing up

To blow up a variety $X \subseteq \mathbb{A}^n$ at a point P:

Blowing up

To blow up a variety $X \subseteq \mathbb{A}^n$ at a point P:

- First, determine all directions through P that lie in the variety. Note that the set of all directions through P in $\mathbb{A}^n$ is $\mathbb{P}^{n-1}$, so we will get a projective variety V.

Blowing up

To blow up a variety $X \subseteq \mathbb{A}^n$ at a point P:

- First, determine all directions through P that lie in the variety. Note that the set of all directions through P in $\mathbb{A}^n$ is $\mathbb{P}^{n-1}$, so we will get a projective variety V.
- Then replace P with the variety V. This will give a new variety X', which is the blowup of X at P.

Blowing up

Algebraically, this can be done by taking a subvariety of $\mathbb{A}^n \times \mathbb{P}^{n-1}$.

Blowing up

Algebraically, this can be done by taking a subvariety of $\mathbb{A}^n \times \mathbb{P}^{n-1}$.

- $\mathbb{P}^{n-1}$ can be thought of as the set of all lines through the point P in $\mathbb{A}^n$. The subset $\{(a, l) \in \mathbb{A}^n \times \mathbb{P}^{n-1} : a \in l\}$ is a subvariety W.
- Remove the point P from X to get $X \setminus P$. Then take the preimage of $X \setminus P$ under the natural projection $W \to \mathbb{A}^n$.
- Finally, take the closure of this preimage to obtain X'.

Blowing up

Algebraically, this can be done by taking a subvariety of $\mathbb{A}^n \times \mathbb{P}^{n-1}$.

- $\mathbb{P}^{n-1}$ can be thought of as the set of all lines through the point P in $\mathbb{A}^n$. The subset $\{(a, l) \in \mathbb{A}^n \times \mathbb{P}^{n-1} : a \in l\}$ is a subvariety W.
- Remove the point P from X to get $X \setminus P$. Then take the preimage of $X \setminus P$ under the natural projection $W \to \mathbb{A}^n$.
- Finally, take the closure of this preimage to obtain X'.

This increases the dimension of the ambient space, and it means you have to work with projective coordinates. For this reason, it's easier practically to divide up X' into affine patches, which can be embedded in $\mathbb{A}^n$.

Blowing up

Algebraically, this can be done by taking a subvariety of $\mathbb{A}^n \times \mathbb{P}^{n-1}$.

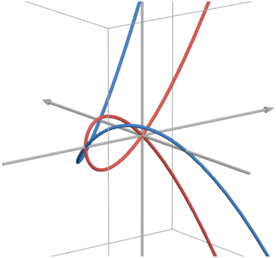
- $\mathbb{P}^{n-1}$ can be thought of as the set of all lines through the point P in $\mathbb{A}^n$. The subset $\{(a, l) \in \mathbb{A}^n \times \mathbb{P}^{n-1} : a \in l\}$ is a subvariety W.
- Remove the point P from X to get $X \setminus P$. Then take the preimage of $X \setminus P$ under the natural projection $W \to \mathbb{A}^n$.
- Finally, take the closure of this preimage to obtain X'.

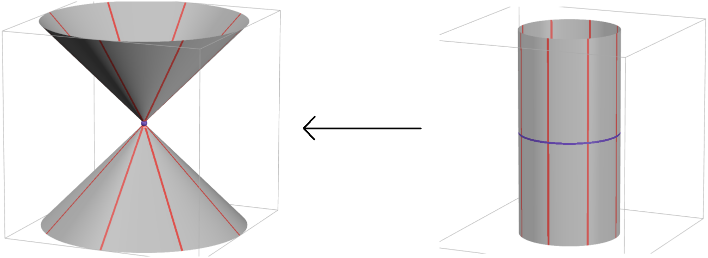
This increases the dimension of the ambient space, and it means you have to work with projective coordinates. For this reason, it's easier practically to divide up X' into affine patches, which can be embedded in $\mathbb{A}^n$.
For example, in $\mathbb{A}^3$, one of the affine patches is obtained by substituting in

$$x' = x, \quad y' = xy, \quad z' = xz.$$

The other two patches are similar.

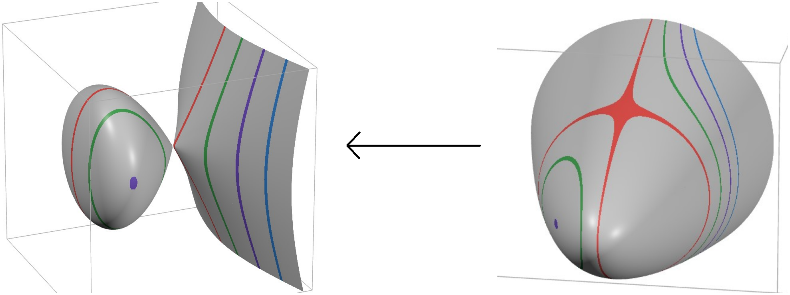
Blowing up - examples





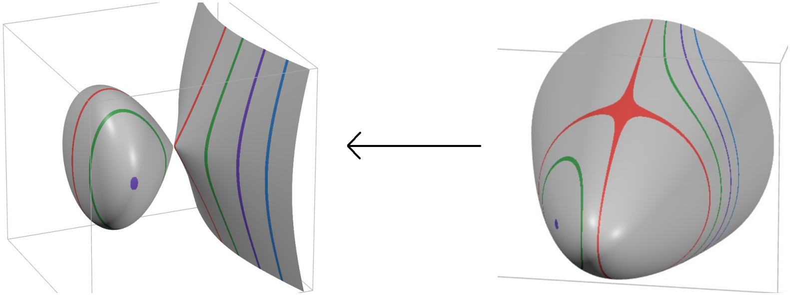
Back to our example

When we blow up our elliptic surface $\mathcal{E}$, we obtain the following surface $\mathcal{E}'$:



Back to our example

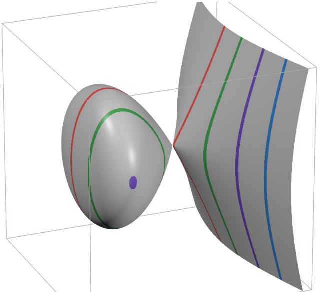
When we blow up our elliptic surface $\mathcal{E}$, we obtain the following surface $\mathcal{E}'$:

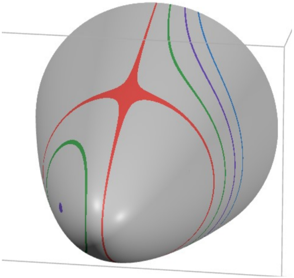


The fibres above t for $t \neq 0$ remain unchanged (we have $E_t \cong E'_t$.)

Back to our example

When we blow up our elliptic surface $\mathcal{E}$, we obtain the following surface $\mathcal{E}'$:





The fibres above t for $t \neq 0$ remain unchanged (we have $E_t \cong E'_t$.)
The fibre above $t = 0$, on the other hand, now looks like the intersection of two curves, each isomorphic to $\mathbb{P}^1$.

Reduction type

There are many different types of singularity one can get when writing down a Weierstrass equation for an elliptic surface, and each will give a different fibre.

Reduction type

There are many different types of singularity one can get when writing down a Weierstrass equation for an elliptic surface, and each will give a different fibre. We've seen three types of fibre so far:

Reduction type

There are many different types of singularity one can get when writing down a Weierstrass equation for an elliptic surface, and each will give a different fibre. We've seen three types of fibre so far:

- I_0: An elliptic curve (a good fibre).

Reduction type

There are many different types of singularity one can get when writing down a Weierstrass equation for an elliptic surface, and each will give a different fibre. We've seen three types of fibre so far:

- I_0: An elliptic curve (a good fibre).
- I_1: A nodal cubic.

Reduction type

There are many different types of singularity one can get when writing down a Weierstrass equation for an elliptic surface, and each will give a different fibre. We've seen three types of fibre so far:

- I_0: An elliptic curve (a good fibre).
- I_1: A nodal cubic.
- I_2: Two copies of $\mathbb{P}^1$, intersecting transversely at 2 points.

Reduction type

There are many different types of singularity one can get when writing down a Weierstrass equation for an elliptic surface, and each will give a different fibre. We've seen three types of fibre so far:

- I_0: An elliptic curve (a good fibre).
- I_1: A nodal cubic.
- I_2: Two copies of $\mathbb{P}^1$, intersecting transversely at 2 points.

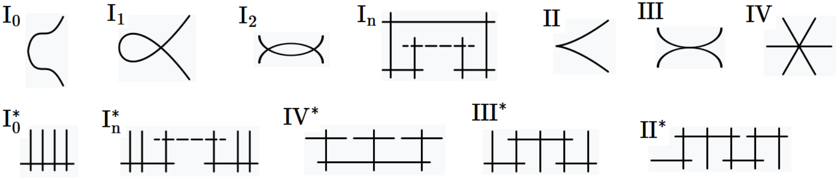
Kodaira classified all possible fibres of minimal elliptic surfaces, and gave each a *Kodaira symbol*.

Reduction type

There are many different types of singularity one can get when writing down a Weierstrass equation for an elliptic surface, and each will give a different fibre. We've seen three types of fibre so far:

- I_0: An elliptic curve (a good fibre).
- I_1: A nodal cubic.
- I_2: Two copies of $\mathbb{P}^1$, intersecting transversely at 2 points.

Kodaira classified all possible fibres of minimal elliptic surfaces, and gave each a *Kodaira symbol*.



Tate's algorithm

To figure out which reduction type you get once you resolve all singularities on a fibre above $t_0 \in \mathbb{P}^1$, you would need to perform the sequence of blowups.

Tate's algorithm

To figure out which reduction type you get once you resolve all singularities on a fibre above $t_0 \in \mathbb{P}^1$, you would need to perform the sequence of blowups.
Tate came up with an algorithm to describe how to do the blowups. It looks something like this:

Tate's algorithm

To figure out which reduction type you get once you resolve all singularities on a fibre above $t_0 \in \mathbb{P}^1$, you would need to perform the sequence of blowups.

Tate came up with an algorithm to describe how to do the blowups. It looks something like this:

- Step 0: Make your equation for the surface *integral* at t_0. In other words, perform a change of coordinates so that each $v(a_i) \geq 0$. (Here v denotes the order of vanishing at t_0.)

Tate's algorithm

To figure out which reduction type you get once you resolve all singularities on a fibre above $t_0 \in \mathbb{P}^1$, you would need to perform the sequence of blowups.
Tate came up with an algorithm to describe how to do the blowups. It looks something like this:

- Step 0: Make your equation for the surface *integral* at t_0. In other words, perform a change of coordinates so that each $v(a_i) \geq 0$. (Here v denotes the order of vanishing at t_0.)
- Step 1: If $v(\Delta) = 0$ then the fibre is smooth and we have type I_0. Otherwise go to the next step.

Tate's algorithm

To figure out which reduction type you get once you resolve all singularities on a fibre above $t_0 \in \mathbb{P}^1$, you would need to perform the sequence of blowups.

Tate came up with an algorithm to describe how to do the blowups. It looks something like this:

- Step 0: Make your equation for the surface *integral* at t_0. In other words, perform a change of coordinates so that each $v(a_i) \geq 0$. (Here v denotes the order of vanishing at t_0.)
- Step 1: If $v(\Delta) = 0$ then the fibre is smooth and we have type I_0. Otherwise go to the next step.
- Step 2-10: Do a blowup. If (some condition on $v(a_i)$) then we have type (something). Otherwise go to the next step.

Tate's algorithm

To figure out which reduction type you get once you resolve all singularities on a fibre above $t_0 \in \mathbb{P}^1$, you would need to perform the sequence of blowups.
Tate came up with an algorithm to describe how to do the blowups. It looks something like this:

- Step 0: Make your equation for the surface *integral* at t_0. In other words, perform a change of coordinates so that each $v(a_i) \geq 0$. (Here v denotes the order of vanishing at t_0.)
- Step 1: If $v(\Delta) = 0$ then the fibre is smooth and we have type I_0. Otherwise go to the next step.
- Step 2-10: Do a blowup. If (some condition on $v(a_i)$) then we have type (something). Otherwise go to the next step.
- Step 11: If you've reached this point then your equation was not minimal. Make a change of coordinates to decrease $v(\Delta)$ and go back to Step 1.

Tate's algorithm

Tate did the blowups so you don't have to. As it turns out, if the residue characteristic is not 2 or 3, you can deduce the reduction type solely from the values of $v(\Delta)$ and $v(j)$.

Tate's algorithm

Tate did the blowups so you don't have to. As it turns out, if the residue characteristic is not 2 or 3, you can deduce the reduction type solely from the values of $v(\Delta)$ and $v(j)$.

Type	I_0	I_n	II	III	IV	I_0^*	I_n^*	IV^*	III^*	II^*
$v(j)$	≥ 0	$-n$	≥ 0	≥ 0	≥ 0	≥ 0	$-n$	≥ 0	≥ 0	≥ 0
$v(\Delta)$	0	n	2	3	4	6	$n+6$	8	9	10

Tate's algorithm

Tate did the blowups so you don't have to. As it turns out, if the residue characteristic is not 2 or 3, you can deduce the reduction type solely from the values of $v(\Delta)$ and $v(j)$.

Type	I_0	I_n	II	III	IV	I_0^*	I_n^*	IV^*	III^*	II^*
$v(j)$	≥ 0	$-n$	≥ 0	≥ 0	≥ 0	≥ 0	$-n$	≥ 0	≥ 0	≥ 0
$v(\Delta)$	0	n	2	3	4	6	$n+6$	8	9	10

Note: even if your Weierstrass equation might not be minimal, you still know $v(\Delta) \mod 12$, so you have enough information to determine the reduction type.

Tate's algorithm

Tate did the blowups so you don't have to. As it turns out, if the residue characteristic is not 2 or 3, you can deduce the reduction type solely from the values of $v(\Delta)$ and $v(j)$.

Type	I_0	I_n	II	III	IV	I_0^*	I_n^*	IV^*	III^*	II^*
$v(j)$	≥ 0	$-n$	≥ 0	≥ 0	≥ 0	≥ 0	$-n$	≥ 0	≥ 0	≥ 0
$v(\Delta)$	0	n	2	3	4	6	$n+6$	8	9	10

Note: even if your Weierstrass equation might not be minimal, you still know $v(\Delta) \mod 12$, so you have enough information to determine the reduction type.
However, if you want explicit formulae for the smooth surface, you do have to do all the blowups.

Section 2

The Brauer group

Cyclic algebras

The Brauer group of a field K can be thought of as the group $\mathrm{Br}(K)$ of central simple algebras over K, or as the Galois cohomology group $H^2(K, \bar{K}^\times)$.

Cyclic algebras

The Brauer group of a field K can be thought of as the group $\mathrm{Br}(K)$ of central simple algebras over K, or as the Galois cohomology group $H^2(K, \bar{K}^\times)$.

One simple example of a central simple algebra is a *cyclic algebra*.

Cyclic algebras

The Brauer group of a field K can be thought of as the group $\mathrm{Br}(K)$ of central simple algebras over K, or as the Galois cohomology group $H^2(K, \bar{K}^\times)$.

One simple example of a central simple algebra is a *cyclic algebra*.

Definiton (Cyclic algebra)

Let K be a field containing a (fixed) primitive nth root of unity ζ_n.
Let $a, b \in K^\times$. Then the *cyclic algebra* $(a, b)_n$ is the K-algebra generated by i, j satisfying

$$i^n = a, \quad j^n = b, \quad ij = \zeta_n ji.$$

Cyclic algebras

This can also be understood in terms of Galois cohomology.

Cyclic algebras

This can also be understood in terms of Galois cohomology.

- We may identify $\mathbb{Z}/n\mathbb{Z} \cong \mu_n$ by sending 1 to ζ_n.

Cyclic algebras

This can also be understood in terms of Galois cohomology.

- We may identify $\mathbb{Z}/n\mathbb{Z} \cong \mu_n$ by sending 1 to ζ_n.
- Then, by Kummer theory, $H^1(K, \mathbb{Z}/n\mathbb{Z}) \cong K^\times/(K^\times)^n$. So we can think of $a, b \in K^\times$ as representing cocycle classes in $H^1(K, \mathbb{Z}/n\mathbb{Z})$.

Cyclic algebras

This can also be understood in terms of Galois cohomology.

- We may identify $\mathbb{Z}/n\mathbb{Z} \cong \mu_n$ by sending 1 to ζ_n.
- Then, by Kummer theory, $H^1(K, \mathbb{Z}/n\mathbb{Z}) \cong K^\times/(K^\times)^n$. So we can think of $a, b \in K^\times$ as representing cocycle classes in $H^1(K, \mathbb{Z}/n\mathbb{Z})$.
- The tensor product $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$ is just $\mathbb{Z}/n\mathbb{Z}$. We can then identify this with $\mu_n \subseteq \bar{K}^\times$.

Cyclic algebras

This can also be understood in terms of Galois cohomology.

- We may identify $\mathbb{Z}/n\mathbb{Z} \cong \mu_n$ by sending 1 to ζ_n.
- Then, by Kummer theory, $H^1(K, \mathbb{Z}/n\mathbb{Z}) \cong K^\times/(K^\times)^n$. So we can think of $a, b \in K^\times$ as representing cocycle classes in $H^1(K, \mathbb{Z}/n\mathbb{Z})$.
- The tensor product $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$ is just $\mathbb{Z}/n\mathbb{Z}$. We can then identify this with $\mu_n \subseteq \bar{K}^\times$.
- This induces a cup-product map

$$H^1(K, \mathbb{Z}/n\mathbb{Z}) \otimes H^1(K, \mathbb{Z}/n\mathbb{Z}) \to H^2(K, \bar{K}^\times).$$

Cyclic algebras

This can also be understood in terms of Galois cohomology.

- We may identify $\mathbb{Z}/n\mathbb{Z} \cong \mu_n$ by sending 1 to ζ_n.
- Then, by Kummer theory, $H^1(K, \mathbb{Z}/n\mathbb{Z}) \cong K^\times/(K^\times)^n$. So we can think of $a, b \in K^\times$ as representing cocycle classes in $H^1(K, \mathbb{Z}/n\mathbb{Z})$.
- The tensor product $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$ is just $\mathbb{Z}/n\mathbb{Z}$. We can then identify this with $\mu_n \subseteq \bar{K}^\times$.
- This induces a cup-product map

$$H^1(K, \mathbb{Z}/n\mathbb{Z}) \otimes H^1(K, \mathbb{Z}/n\mathbb{Z}) \to H^2(K, \bar{K}^\times).$$

- The cup product of a with b is then the class corresponding to $(a, b)_n$.

The Brauer group of a variety

The *Brauer group* of X is defined to be the étale cohomology group $H^2(X, \mathbb{G}_m)$.

The Brauer group of a variety

The *Brauer group* of X is defined to be the étale cohomology group $H^2(X, \mathbb{G}_m)$. Suppose X is smooth and integral, and let $k(X)$ be its function field.

The Brauer group of a variety

The *Brauer group* of X is defined to be the étale cohomology group $H^2(X, \mathbb{G}_m)$. Suppose X is smooth and integral, and let $k(X)$ be its function field.
Then we have an injective map

$$\mathrm{Br}(X) \to \mathrm{Br}(k(X))$$

so we can think of $\mathrm{Br}(X)$ as a subgroup of $\mathrm{Br}(k(X))$.

The Brauer group of a variety

The *Brauer group* of X is defined to be the étale cohomology group $H^2(X, \mathbb{G}_m)$. Suppose X is smooth and integral, and let $k(X)$ be its function field.
Then we have an injective map

$$\mathrm{Br}(X) \to \mathrm{Br}(k(X))$$

so we can think of $\mathrm{Br}(X)$ as a subgroup of $\mathrm{Br}(k(X))$.
By functoriality of cohomology, we can evaluate an element $A \in \mathrm{Br}(X)$ at any point $P \in X$. This is not true of elements of $\mathrm{Br}(k(X))$ in general.

Brauer group of an elliptic curve

Let E be an elliptic curve defined over K.

Brauer group of an elliptic curve

Let E be an elliptic curve defined over K. Then $\mathrm{Br}(E)$ is well understood: we have

$$\mathrm{Br}(E) \cong \mathrm{Br}^0(E) \oplus \mathrm{Br}(K)$$

Brauer group of an elliptic curve

Let E be an elliptic curve defined over K. Then $\mathrm{Br}(E)$ is well understood: we have

$$\mathrm{Br}(E) \cong \mathrm{Br}^0(E) \oplus \mathrm{Br}(K)$$

where $\mathrm{Br}^0(E)$ is the subgroup of elements that evaluate to zero at the identity point $O_E \in E$.

Brauer group of an elliptic curve

Let E be an elliptic curve defined over K. Then $\mathrm{Br}(E)$ is well understood: we have

$$\mathrm{Br}(E) \cong \mathrm{Br}^0(E) \oplus \mathrm{Br}(K)$$

where $\mathrm{Br}^0(E)$ is the subgroup of elements that evaluate to zero at the identity point $O_E \in E$. Moreover, we have an isomorphism

$$\mathrm{Br}^0(E) \cong H^1(K, E).$$

Descent by isogeny

The group $H^1(K, E)$ is still quite large, so we look at smaller, more manageable subgroups.

Descent by isogeny

The group $H^1(K, E)$ is still quite large, so we look at smaller, more manageable subgroups. Let $\phi : E \to E'$ be an isogeny, and $\hat{\phi} : E' \to E$ the dual isogeny.

Descent by isogeny

The group $H^1(K, E)$ is still quite large, so we look at smaller, more manageable subgroups. Let $\phi : E \to E'$ be an isogeny, and $\hat{\phi} : E' \to E$ the dual isogeny.

- There is a short exact sequence

$$0 \to E[\phi] \to E \to E' \to 0.$$

Descent by isogeny

The group $H^1(K, E)$ is still quite large, so we look at smaller, more manageable subgroups. Let $\phi : E \to E'$ be an isogeny, and $\hat{\phi} : E' \to E$ the dual isogeny.

- There is a short exact sequence

$$0 \to E[\phi] \to E \to E' \to 0.$$

- This gives a long exact sequence

$$\cdots \to E'(K) \to H^1(K, E[\phi]) \to H^1(k, E) \to \dots.$$

Descent by isogeny

The group $H^1(K, E)$ is still quite large, so we look at smaller, more manageable subgroups. Let $\phi : E \to E'$ be an isogeny, and $\hat{\phi} : E' \to E$ the dual isogeny.

- There is a short exact sequence

$$0 \to E[\phi] \to E \to E' \to 0.$$

- This gives a long exact sequence

$$\cdots \to E'(K) \to H^1(K, E[\phi]) \to H^1(k, E) \to \ldots.$$

Descent by isogeny

$$\cdots \to E'(K) \to H^1(K, E[\phi]) \to H^1(k, E) \to \ldots.$$

Descent by isogeny

$$\cdots \to E'(K) \to H^1(K, E[\phi]) \to H^1(k, E) \to \dots .$$

Suppose K contains a primitive nth root of unity.

Descent by isogeny

$$\cdots \to E'(K) \to H^1(K, E[\phi]) \to H^1(k, E) \to \ldots.$$

Suppose K contains a primitive nth root of unity.
If we choose an point $P \in E(K)$ of order n, and set ϕ to be the isogeny with kernel generated by P, then we will have

Descent by isogeny

$$\cdots \to E'(K) \to H^1(K, E[\phi]) \to H^1(k, E) \to \ldots.$$

Suppose K contains a primitive nth root of unity.
If we choose an point $P \in E(K)$ of order n, and set ϕ to be the isogeny with kernel generated by P, then we will have

$$H^1(K, E[\phi]) \cong H^1(K, \mathbb{Z}/n\mathbb{Z}) \cong K^\times/(K^\times)^n.$$

Therefore, elements of $K^\times/(K^\times)^n$ will give us elements of $H^1(K, E)$, and hence elements of the Brauer group $\mathrm{Br}(E)$.

Brauer elements of an elliptic curve

Next, we need to describe the map $K^\times/(K^\times)^n \to \mathrm{Br}(E)$ explicitly.

Brauer elements of an elliptic curve

Next, we need to describe the map $K^\times/(K^\times)^n \to \mathrm{Br}(E)$ explicitly. To do this, we use the following theorem:

Theorem

We have a compatibility between pairings

$$\begin{array}{ccccc} H^1(K,E) & \times & E(K) & \longrightarrow & \mathrm{Br}(K) \\ \uparrow & & \downarrow & & \| \\ H^1(K,E[\phi]) & \times & H^1(K,E'[\hat{\phi}]) & \longrightarrow & \mathrm{Br}(K) \end{array}$$

Brauer elements of an elliptic curve

Next, we need to describe the map $K^\times/(K^\times)^n \to \mathrm{Br}(E)$ explicitly. To do this, we use the following theorem:

Theorem

We have a compatibility between pairings

$$\begin{array}{ccccc} H^1(K,E) & \times & E(K) & \longrightarrow & \mathrm{Br}(K) \\ \uparrow & & \downarrow & & \| \\ H^1(K,E[\phi]) & \times & H^1(K,E'[\hat{\phi}]) & \longrightarrow & \mathrm{Br}(K) \end{array}$$

The top pairing is given by evaluating the Brauer element corrresponding to $C \in H^1(K,E)$ at $P \in E(K)$.
The bottom pairing is given by cup-product and the Weil pairing $E[\phi] \times E[\hat{\phi}] \to \mu_n$.

Brauer elements of an elliptic curve

$$\begin{array}{ccccc} H^1(K,E) & \times & E(K) & \longrightarrow & \mathrm{Br}(K) \\ \uparrow & & \downarrow & & \| \\ K^\times/(K^\times)^n & \times & K^\times/(K^\times)^n & \longrightarrow & \mathrm{Br}(K) \end{array}$$

Brauer elements of an elliptic curve

$$\begin{array}{ccccc} H^1(K,E) & \times & E(K) & \longrightarrow & \mathrm{Br}(K) \\ \uparrow & & \downarrow & & \| \\ K^\times/(K^\times)^n & \times & K^\times/(K^\times)^n & \longrightarrow & \mathrm{Br}(K) \end{array}$$

The map $E(K) \to K^\times/(K^\times)^n$ in the middle column can be described by some function $f \in K(E)$.

Brauer elements of an elliptic curve

$$\begin{array}{ccccc} H^1(K,E) & \times & E(K) & \longrightarrow & \mathrm{Br}(K) \\ \uparrow & & \downarrow & & \| \\ K^\times/(K^\times)^n & \times & K^\times/(K^\times)^n & \longrightarrow & \mathrm{Br}(K) \end{array}$$

The map $E(K) \to K^\times/(K^\times)^n$ in the middle column can be described by some function $f \in K(E)$.
So if we fix some $a \in K^\times/(K^\times)^n$, and let A be its image under the map $K^\times/(K^\times)^n \cong H^1(K, E[\phi]) \to \mathrm{Br}(E)$, we get

Brauer elements of an elliptic curve

$$\begin{array}{ccccc} H^1(K,E) & \times & E(K) & \longrightarrow & \mathrm{Br}(K) \\ \uparrow & & \downarrow & & \| \\ K^\times/(K^\times)^n & \times & K^\times/(K^\times)^n & \longrightarrow & \mathrm{Br}(K) \end{array}$$

The map $E(K) \to K^\times/(K^\times)^n$ in the middle column can be described by some function $f \in K(E)$.
So if we fix some $a \in K^\times/(K^\times)^n$, and let A be its image under the map $K^\times/(K^\times)^n \cong H^1(K, E[\phi]) \to \mathrm{Br}(E)$, we get

$$A(P) = (a, f(P))_n.$$

Brauer elements of an elliptic curve

$$A(P) = (a, f(P))_n.$$

Brauer elements of an elliptic curve

$$A(P) = (a, f(P))_n.$$

This suggests that the element $A \in \mathrm{Br}(E) \subseteq \mathrm{Br}(K(E))$ is given by the cyclic algebra $(a, f)_n$.

Brauer elements of an elliptic curve

$$A(P) = (a, f(P))_n.$$

This suggests that the element $A \in \mathrm{Br}(E) \subseteq \mathrm{Br}(K(E))$ is given by the cyclic algebra $(a, f)_n$.

Indeed, the above formula still holds upon extending K to a larger field, so one can prove this by substituting in P to be the generic point of E. So we have

$$A = (a, f)_n.$$

The residue map

We would now like a description of $\mathrm{Br}(X)$ for X a more general variety.

The residue map

We would now like a description of $\mathrm{Br}(X)$ for X a more general variety. Let D be an irreducible codimension-1 subvariety of X, i.e. an irreducible divisor.

The residue map

We would now like a description of $\mathrm{Br}(X)$ for X a more general variety. Let D be an irreducible codimension-1 subvariety of X, i.e. an irreducible divisor. It has an associated discrete valuation $v_D : k(X)^\times \to \mathbb{Z}$.

The residue map

We would now like a description of $\mathrm{Br}(X)$ for X a more general variety. Let D be an irreducible codimension-1 subvariety of X, i.e. an irreducible divisor. It has an associated discrete valuation $v_D : k(X)^\times \to \mathbb{Z}$. One can then use this to define a *residue map*

$$\partial_D : \mathrm{Br}(k(X)) \to H^1(k(D), \mathbb{Q}/\mathbb{Z}).$$

The residue map

We would now like a description of $\mathrm{Br}(X)$ for X a more general variety. Let D be an irreducible codimension-1 subvariety of X, i.e. an irreducible divisor. It has an associated discrete valuation $v_D : k(X)^\times \to \mathbb{Z}$. One can then use this to define a *residue map*

$$\partial_D : \mathrm{Br}(k(X)) \to H^1(k(D), \mathbb{Q}/\mathbb{Z}).$$

Theorem

The kernel of ∂_D is the set of all elements of $\mathrm{Br}(k(X))$ that can be evaluated at D.

The residue map

We would now like a description of $\mathrm{Br}(X)$ for X a more general variety. Let D be an irreducible codimension-1 subvariety of X, i.e. an irreducible divisor. It has an associated discrete valuation $v_D : k(X)^\times \to \mathbb{Z}$. One can then use this to define a *residue map*

$$\partial_D : \mathrm{Br}(k(X)) \to H^1(k(D), \mathbb{Q}/\mathbb{Z}).$$

Theorem

The kernel of ∂_D is the set of all elements of $\mathrm{Br}(k(X))$ that can be evaluated at D.

Theorem

Let X be a smooth, proper, integral variety over a field of characteristic 0. Then $A \in \mathrm{Br}(k(X))$ lies in $\mathrm{Br}(X)$ if and only if $\partial_D(A) = 0$ for all irreducible divisors D of X.

Brauer group of an elliptic surface

For $X = \mathcal{E}$ an elliptic surface, there are lots of irreducible divisors.

Brauer group of an elliptic surface

For $X = \mathcal{E}$ an elliptic surface, there are lots of irreducible divisors. However, they come in two types:

Brauer group of an elliptic surface

For $X = \mathcal{E}$ an elliptic surface, there are lots of irreducible divisors. However, they come in two types:

- Horizontal divisors: these correspond to closed points of the generic fibre E, an elliptic curve defined over $K = k(t)$.

Brauer group of an elliptic surface

For $X = \mathcal{E}$ an elliptic surface, there are lots of irreducible divisors. However, they come in two types:

- Horizontal divisors: these correspond to closed points of the generic fibre E, an elliptic curve defined over $K = k(t)$.
- Vertical divisors: these are irreducible components of fibres of the fibration.

Therefore, all elements of $\mathrm{Br}(E) \subseteq \mathrm{Br}(k(\mathcal{E}))$ automatically have zero residue on all horizontal divisors of $\mathcal{E}$.

Brauer group of an elliptic surface

For $X = \mathcal{E}$ an elliptic surface, there are lots of irreducible divisors. However, they come in two types:

- Horizontal divisors: these correspond to closed points of the generic fibre E, an elliptic curve defined over $K = k(t)$.
- Vertical divisors: these are irreducible components of fibres of the fibration.

Therefore, all elements of $\mathrm{Br}(E) \subseteq \mathrm{Br}(k(\mathcal{E}))$ automatically have zero residue on all horizontal divisors of $\mathcal{E}$. And so we only need to check their residues for vertical divisors.

Local conditions

Let's split up the checking of residues fibre by fibre.

Local conditions

Let's split up the checking of residues fibre by fibre.
We pick a point $t \in \mathbb{P}^1$. Then the fibre $\pi^{-1}(t)$ will be one of the Kodaira reduction types seen earlier.

Local conditions

Let's split up the checking of residues fibre by fibre.
We pick a point $t \in \mathbb{P}^1$. Then the fibre $\pi^{-1}(t)$ will be one of the Kodaira reduction types seen earlier.
Each irreducible component $D \subseteq \pi^{-1}(t)$ gives a residue map ∂_D to check.

Local conditions

Let's split up the checking of residues fibre by fibre.
We pick a point $t \in \mathbb{P}^1$. Then the fibre $\pi^{-1}(t)$ will be one of the Kodaira reduction types seen earlier.
Each irreducible component $D \subseteq \pi^{-1}(t)$ gives a residue map ∂_D to check.
We can convert the condition

$$\partial_D((a, f)_n) = 0$$

into a condition on $a \in K^\times = k(t)^\times$.

Local conditions

Let's split up the checking of residues fibre by fibre.
We pick a point $t \in \mathbb{P}^1$. Then the fibre $\pi^{-1}(t)$ will be one of the Kodaira reduction types seen earlier.
Each irreducible component $D \subseteq \pi^{-1}(t)$ gives a residue map ∂_D to check.
We can convert the condition

$$\partial_D((a, f)_n) = 0$$

into a condition on $a \in K^\times = k(t)^\times$.
The combination of all these conditions will be the *local condition* on $a \in K^\times$ given by P.

Examples

Local conditions

Examples

- If the fibre above P has type I_0, then the local condition is

$$\mathrm{ord}_P(a) \equiv 0 \mod n.$$

Local conditions

Examples

- If the fibre above P has type I_0, then the local condition is

$$\mathrm{ord}_P(a) \equiv 0 \mod n.$$

- If the fibre above P has type I_1, then the local condition is trivial (everything satisfies the condition).

Local conditions

Examples

- If the fibre above P has type I_0, then the local condition is

$$\mathrm{ord}_P(a) \equiv 0 \mod n.$$

- If the fibre above P has type I_1, then the local condition is trivial (everything satisfies the condition).

We will call the subgroup of $K^\times/(K^\times)^n$ satisfying all the local conditions the *Selmer group*.

Local conditions

Examples

- If the fibre above P has type I_0, then the local condition is

$$\mathrm{ord}_P(a) \equiv 0 \mod n.$$

- If the fibre above P has type I_1, then the local condition is trivial (everything satisfies the condition).

We will call the subgroup of $K^\times/(K^\times)^n$ satisfying all the local conditions the *Selmer group*. It consists of the elements that map to things inside $\mathrm{Br}(\mathcal{E})$ under the map $K^\times/(K^\times)^n \to \mathrm{Br}(E)$.

Brauer-Manin obstruction to weak approximation

Brauer-Manin obstruction to weak approximation

To find a Brauer-Manin obstruction to weak approximation for our surface $\mathcal{E}$, it suffices to find some $A \in \mathrm{Br}(\mathcal{E})$ and some adelic point $P \in \mathcal{E}(\mathbb{A}_K)$ such that $\sum_v \mathrm{inv}\, A(P_v) \neq 0$.

Brauer-Manin obstruction to weak approximation

To find a Brauer-Manin obstruction to weak approximation for our surface $\mathcal{E}$, it suffices to find some $A \in \mathrm{Br}(\mathcal{E})$ and some adelic point $P \in \mathcal{E}(\mathbb{A}_K)$ such that $\sum_v \operatorname{inv} A(P_v) \neq 0$.
Our method of constructing elements of $\mathrm{Br}(\mathcal{E})$ always give elements in $\mathrm{Br}^0(E)$. So they will always evaluate to 0 on the identity $O_E \in E$.

Brauer-Manin obstruction to weak approximation

To find a Brauer-Manin obstruction to weak approximation for our surface $\mathcal{E}$, it suffices to find some $A \in \mathrm{Br}(\mathcal{E})$ and some adelic point $P \in \mathcal{E}(\mathbb{A}_K)$ such that $\sum_v \operatorname{inv} A(P_v) \neq 0$.

Our method of constructing elements of $\mathrm{Br}(\mathcal{E})$ always give elements in $\mathrm{Br}^0(E)$. So they will always evaluate to 0 on the identity $O_E \in E$.

This tells us we can always find local points $P_v \in \mathcal{E}(K_v)$ with $\operatorname{inv} A(P_v) = 0$.

Brauer-Manin obstruction to weak approximation

To find a Brauer-Manin obstruction to weak approximation for our surface $\mathcal{E}$, it suffices to find some $A \in \mathrm{Br}(\mathcal{E})$ and some adelic point $P \in \mathcal{E}(\mathbb{A}_K)$ such that $\sum_v \operatorname{inv} A(P_v) \neq 0$.
Our method of constructing elements of $\mathrm{Br}(\mathcal{E})$ always give elements in $\mathrm{Br}^0(E)$. So they will always evaluate to 0 on the identity $O_E \in E$.
This tells us we can always find local points $P_v \in \mathcal{E}(K_v)$ with $\operatorname{inv} A(P_v) = 0$. So it suffices to find a local point P_v such that $\operatorname{inv} A(P_v) \neq 0$.

Example of a Brauer-Manin obstruction

Let $k = \mathbb{Q}(\zeta_3)$, and let $\mathcal{E}$ be the elliptic surface given by equation

$$y^2 + xy + \frac{1}{27}t^4 y = x^3.$$

Example of a Brauer-Manin obstruction

Let $k = \mathbb{Q}(\zeta_3)$, and let $\mathcal{E}$ be the elliptic surface given by equation

$$y^2 + xy + \frac{1}{27}t^4 y = x^3.$$

- The section $(x = 0, y = 0, t)$ gives a point of order 3 on the generic fibre E. So we get a 3-isogeny ϕ.

Example of a Brauer-Manin obstruction

Let $k = \mathbb{Q}(\zeta_3)$, and let $\mathcal{E}$ be the elliptic surface given by equation

$$y^2 + xy + \frac{1}{27}t^4 y = x^3.$$

- The section $(x = 0, y = 0, t)$ gives a point of order 3 on the generic fibre E. So we get a 3-isogeny ϕ.
- The corresponding function $f : E(K) \to K^\times/(K^\times)^n$ is given by $y \in K(E)$.

Example of a Brauer-Manin obstruction

Let $k = \mathbb{Q}(\zeta_3)$, and let $\mathcal{E}$ be the elliptic surface given by equation

$$y^2 + xy + \frac{1}{27}t^4 y = x^3.$$

- The section $(x = 0, y = 0, t)$ gives a point of order 3 on the generic fibre E. So we get a 3-isogeny ϕ.
- The corresponding function $f : E(K) \to K^\times/(K^\times)^n$ is given by $y \in K(E)$.
- The bad fibres have reduction types

0	1	-1	$\pm i$	∞
I_{12}	I_1	I_1	I_1	IV^*

Example of a Brauer-Manin obstruction

The local conditions are:

Example of a Brauer-Manin obstruction

The local conditions are:

- $\operatorname{ord}_t(a) \equiv 0 \mod 3$ for all t except for $0, \pm 1, \pm i, \infty$.

Example of a Brauer-Manin obstruction

The local conditions are:

- $\text{ord}_t(a) \equiv 0 \mod 3$ for all t except for $0, \pm 1, \pm i, \infty$.
- $\text{ord}_t(a) \equiv 0 \mod 3$ and $a(t)$ is a perfect cube for $t = 0, \infty$.

Example of a Brauer-Manin obstruction

The local conditions are:

- $\mathrm{ord}_t(a) \equiv 0 \mod 3$ for all t except for $0, \pm 1, \pm i, \infty$.
- $\mathrm{ord}_t(a) \equiv 0 \mod 3$ and $a(t)$ is a perfect cube for $t = 0, \infty$.
- No condition for $t = \pm 1, \pm i$.

Example of a Brauer-Manin obstruction

The local conditions are:

- $\mathrm{ord}_t(a) \equiv 0 \mod 3$ for all t except for $0, \pm 1, \pm i, \infty$.
- $\mathrm{ord}_t(a) \equiv 0 \mod 3$ and $a(t)$ is a perfect cube for $t = 0, \infty$.
- No condition for $t = \pm 1, \pm i$.

Setting $a = \frac{t-1}{t+1}$ satisfies all of these local conditions.

Example of a Brauer-Manin obstruction

The local conditions are:

- $\text{ord}_t(a) \equiv 0 \mod 3$ for all t except for $0, \pm 1, \pm i, \infty$.
- $\text{ord}_t(a) \equiv 0 \mod 3$ and $a(t)$ is a perfect cube for $t = 0, \infty$.
- No condition for $t = \pm 1, \pm i$.

Setting $a = \frac{t-1}{t+1}$ satisfies all of these local conditions. So

$$A = \left(\frac{t-1}{t+1}, y\right)_3$$

is an element of $\text{Br}(\mathcal{E})$.

Example of a Brauer-Manin obstruction

The local conditions are:

- $\operatorname{ord}_t(a) \equiv 0 \mod 3$ for all t except for $0, \pm 1, \pm i, \infty$.
- $\operatorname{ord}_t(a) \equiv 0 \mod 3$ and $a(t)$ is a perfect cube for $t = 0, \infty$.
- No condition for $t = \pm 1, \pm i$.

Setting $a = \frac{t-1}{t+1}$ satisfies all of these local conditions. So

$$A = \left(\frac{t-1}{t+1}, y \right)_3$$

is an element of $\mathrm{Br}(\mathcal{E})$. Evaluating this at the local point $(t = 1 - \zeta_3, x = 1, y = \ldots) \in \mathbb{Q}_3(\zeta_3)$ gives $\frac{1}{3}$, which is nonzero.

Example of a Brauer-Manin obstruction

The local conditions are:

- $\text{ord}_t(a) \equiv 0 \mod 3$ for all t except for $0, \pm 1, \pm i, \infty$.
- $\text{ord}_t(a) \equiv 0 \mod 3$ and $a(t)$ is a perfect cube for $t = 0, \infty$.
- No condition for $t = \pm 1, \pm i$.

Setting $a = \frac{t-1}{t+1}$ satisfies all of these local conditions. So

$$A = \left(\frac{t-1}{t+1}, y\right)_3$$

is an element of $\text{Br}(\mathcal{E})$. Evaluating this at the local point $(t = 1 - \zeta_3, x = 1, y = \dots) \in \mathbb{Q}_3(\zeta_3)$ gives $\frac{1}{3}$, which is nonzero. So A gives a Brauer-Manin obstruction to weak approximation on $\mathcal{E}$.

Example of a Brauer-Manin obstruction

We can also construct a Brauer element of order 5 with this method.

Example of a Brauer-Manin obstruction

We can also construct a Brauer element of order 5 with this method. Let $k = \mathbb{Q}(\zeta_5)$, and let $\phi = 1 + \zeta_5 + \zeta_5^{-1}$.

Example of a Brauer-Manin obstruction

We can also construct a Brauer element of order 5 with this method.
Let $k = \mathbb{Q}(\zeta_5)$, and let $\phi = 1 + \zeta_5 + \zeta_5^{-1}$.
Let $\mathcal{E}$ be the elliptic surface given by the equation

$$y^2 + (1 - \phi^5 t^2)xy - \phi^5 t^2 y = x^3 - \phi^5 t^2.$$

By a similar method to the previous example, we get that

$$A = \left(\frac{t-1}{t+1}, xy + y - x^2\right)_5$$

is an element of $\mathrm{Br}(\mathcal{E})$.

Example of a Brauer-Manin obstruction

We can also construct a Brauer element of order 5 with this method.
Let $k = \mathbb{Q}(\zeta_5)$, and let $\phi = 1 + \zeta_5 + \zeta_5^{-1}$.
Let $\mathcal{E}$ be the elliptic surface given by the equation

$$y^2 + (1 - \phi^5 t^2)xy - \phi^5 t^2 y = x^3 - \phi^5 t^2.$$

By a similar method to the previous example, we get that

$$A = \left(\frac{t-1}{t+1}, xy + y - x^2 \right)_5$$

is an element of $\mathrm{Br}(\mathcal{E})$. Evaluating this at the local point $(t = 2, x = -\zeta_5, y = \dots) \in \mathbb{Q}_3(\zeta_3)$ gives $\frac{4}{5}$, which is nonzero.

Example of a Brauer-Manin obstruction

We can also construct a Brauer element of order 5 with this method.
Let $k = \mathbb{Q}(\zeta_5)$, and let $\phi = 1 + \zeta_5 + \zeta_5^{-1}$.
Let $\mathcal{E}$ be the elliptic surface given by the equation

$$y^2 + (1 - \phi^5 t^2)xy - \phi^5 t^2 y = x^3 - \phi^5 t^2.$$

By a similar method to the previous example, we get that

$$A = \left(\frac{t-1}{t+1}, xy + y - x^2 \right)_5$$

is an element of $\mathrm{Br}(\mathcal{E})$. Evaluating this at the local point $(t = 2, x = -\zeta_5, y = \ldots) \in \mathbb{Q}_3(\zeta_3)$ gives $\frac{4}{5}$, which is nonzero.
So A gives a Brauer-Manin obstruction to weak approximation on $\mathcal{E}$.

Thanks for listening!