Brauer elements on elliptic surfaces

Harvey Yau

September 7, 2023

Section 1

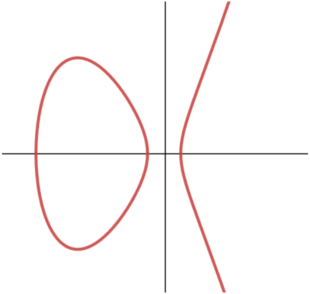
A starter problem

A starter problem

Consider the elliptic curve $E : y^2 = x^3 + 4x^2 - 1$.

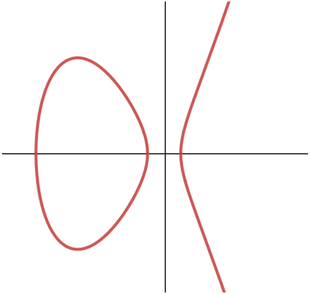
A starter problem

Consider the elliptic curve $E : y^2 = x^3 + 4x^2 - 1$.



A starter problem

Consider the elliptic curve $E : y^2 = x^3 + 4x^2 - 1$.

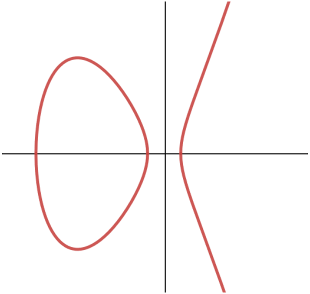


It has lots of rational points on the right component, e.g. $(1, 2)$, but none on the left component.

Is there an explanation for this?

A starter problem

Consider the elliptic curve $E : y^2 = x^3 + 4x^2 - 1$.



It has lots of rational points on the right component, e.g. $(1, 2)$, but none on the left component.

Is there an explanation for this?

Yes! It involves an element of the Brauer group of E...

Section 2

The Brauer group

Central simple algebras

Central simple algebras

Given any field K we can define the *Brauer group* $\mathrm{Br}(K)$.

Central simple algebras

Given any field K we can define the *Brauer group* $\mathrm{Br}(K)$.
Its elements are equivalence classes of *central simple algebras over* K.

Central simple algebras

Given any field K we can define the *Brauer group* $\mathrm{Br}(K)$.
Its elements are equivalence classes of *central simple algebras over K*.

Examples (Central simple algebras)

- $A = M_n(K)$ the matrix algebra of $n \times n$ matrices over K. We call these *trivial*.
- $K = \mathbb{R}, A = \mathbb{H}$ the quaternions.

Central simple algebras

Given any field K we can define the *Brauer group* $\text{Br}(K)$.
Its elements are equivalence classes of *central simple algebras over* K.

Examples (Central simple algebras)

- $A = M_n(K)$ the matrix algebra of $n \times n$ matrices over K.
  We call these *trivial*.
- $K = \mathbb{R}, A = \mathbb{H}$ the quaternions.

We can construct other examples similar to the quaternions...

Quaternion algebras

Quaternion algebras

Let $a, b \in K^\times$.
The *quaternion algebra* $(a, b)_K$ is the K-algebra spanned by $1, i, j, ij$, with multiplication defined by

Quaternion algebras

Let $a, b \in K^\times$.
The *quaternion algebra* $(a, b)_K$ is the K-algebra spanned by $1, i, j, ij$, with multiplication defined by

$$i^2 = a, \quad j^2 = b, \quad ji = -ij.$$

Quaternion algebras

Let $a, b \in K^\times$.
The *quaternion algebra* $(a, b)_K$ is the K-algebra spanned by $1, i, j, ij$, with multiplication defined by

$$i^2 = a, \quad j^2 = b, \quad ji = -ij.$$

Examples

- $(-1, -1)_\mathbb{R}$ is the ordinary quaternions $\mathbb{H}$.

Quaternion algebras

Let $a, b \in K^\times$.
The *quaternion algebra* $(a, b)_K$ is the K-algebra spanned by $1, i, j, ij$, with multiplication defined by

$$i^2 = a, \quad j^2 = b, \quad ji = -ij.$$

Examples

- $(-1, -1)_{\mathbb{R}}$ is the ordinary quaternions $\mathbb{H}$.
- If $a = 1$, $(a, b)_K$ is just the matrix algebra $M_2(K)$.

Examples of Brauer groups

The Brauer group of a field K consists of equivalence classes of central simple algebras over K.

Examples

Examples of Brauer groups

The Brauer group of a field K consists of equivalence classes of central simple algebras over K.

Examples

- If K is algebraically closed, then $\mathrm{Br}(K) = 0$.
  E.g. $\mathrm{Br}(\mathbb{C}) = 0$.

Examples of Brauer groups

The Brauer group of a field K consists of equivalence classes of central simple algebras over K.

Examples

- If K is algebraically closed, then $\mathrm{Br}(K) = 0$.
  E.g. $\mathrm{Br}(\mathbb{C}) = 0$.
- $\mathrm{Br}(\mathbb{R}) \cong \frac{1}{2}\mathbb{Z}/\mathbb{Z}$.

Examples of Brauer groups

The Brauer group of a field K consists of equivalence classes of central simple algebras over K.

Examples

- If K is algebraically closed, then $\mathrm{Br}(K) = 0$.
  E.g. $\mathrm{Br}(\mathbb{C}) = 0$.
- $\mathrm{Br}(\mathbb{R}) \cong \frac{1}{2}\mathbb{Z}/\mathbb{Z}$.
- If K is a non-archimedean local field, then $\mathrm{Br}(K) \cong \mathbb{Q}/\mathbb{Z}$.
  The isomorphism $\mathrm{Br}(K) \to \mathbb{Q}/\mathbb{Z}$ is called the *invariant map* inv_K.

The Brauer group of $\mathbb{Q}$

The Brauer group of $\mathbb{Q}$

For each place v of $\mathbb{Q}$ we get a *local invariant map*

$$\operatorname{Br} \mathbb{Q} \to \operatorname{Br} \mathbb{Q}_v \to \mathbb{Q}/\mathbb{Z}.$$

The Brauer group of $\mathbb{Q}$

For each place v of $\mathbb{Q}$ we get a *local invariant map*

$$\operatorname{Br}\mathbb{Q} \to \operatorname{Br}\mathbb{Q}_v \to \mathbb{Q}/\mathbb{Z}.$$

Theorem (Albert-Brauer-Hasse-Noether)

There is an exact sequence

$$0 \longrightarrow \operatorname{Br}(\mathbb{Q}) \longrightarrow \bigoplus_v \operatorname{Br}(\mathbb{Q}_v) \xrightarrow{\sum \operatorname{inv}_v} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

The Brauer group of $\mathbb{Q}$

For each place v of $\mathbb{Q}$ we get a *local invariant map*

$$\operatorname{Br}\mathbb{Q} \to \operatorname{Br}\mathbb{Q}_v \to \mathbb{Q}/\mathbb{Z}.$$

Theorem (Albert-Brauer-Hasse-Noether)

There is an exact sequence

$$0 \longrightarrow \operatorname{Br}(\mathbb{Q}) \longrightarrow \bigoplus_v \operatorname{Br}(\mathbb{Q}_v) \xrightarrow{\sum \operatorname{inv}_v} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Note that this tells us that the sum of local invariants of any $[A] \in \operatorname{Br}(\mathbb{Q})$ is zero.

The Brauer group of a variety

The Brauer group of a variety

Let X be a nice variety, and let $k(X)$ be its function field.

The Brauer group of a variety

Let X be a nice variety, and let $k(X)$ be its function field.
The *Brauer group* of X consists of the elements of $\mathrm{Br}(k(X))$ that can be evaluated at all (scheme-theoretic) points of X.

Let X be a nice variety, and let $k(X)$ be its function field.
The *Brauer group* of X consists of the elements of $\mathrm{Br}(k(X))$ that can be evaluated at all (scheme-theoretic) points of X.

Examples (Evaluating at a point)

Consider the curve $\mathbb{P}^1_{\mathbb{Q}}$, it has function field $\mathbb{Q}(t)$. Let's take the quaternion algebra $A = (t+3, t-1)_{\mathbb{Q}(t)}$.

The Brauer group of a variety

Let X be a nice variety, and let $k(X)$ be its function field.
The *Brauer group* of X consists of the elements of $\mathrm{Br}(k(X))$ that can be evaluated at all (scheme-theoretic) points of X.

Examples (Evaluating at a point)

Consider the curve $\mathbb{P}^1_{\mathbb{Q}}$, it has function field $\mathbb{Q}(t)$. Let's take the quaternion algebra $A = (t+3, t-1)_{\mathbb{Q}(t)}$.

- We can evaluate A at the point $t = 0$, which gives the quaternion algebra $(3, -1)_{\mathbb{Q}}$.

The Brauer group of a variety

Examples (Evaluating at a point)

Consider the curve $\mathbb{P}^1_{\mathbb{Q}}$, it has function field $\mathbb{Q}(t)$. Let's take the quaternion algebra $A = (t+3, t-1)_{\mathbb{Q}(t)}$.

- We can evaluate A at the point $t = 0$, which gives the quaternion algebra $(3, -1)_{\mathbb{Q}}$.

Examples (Evaluating at a point)

Consider the curve $\mathbb{P}^1_{\mathbb{Q}}$, it has function field $\mathbb{Q}(t)$. Let's take the quaternion algebra $A = (t+3, t-1)_{\mathbb{Q}(t)}$.

- We can evaluate A at the point $t = 0$, which gives the quaternion algebra $(3, -1)_{\mathbb{Q}}$.
- Directly trying to evaluate A at $t = 1$ doesn't work since $t - 1 = 0$ there.

The Brauer group of a variety

Examples (Evaluating at a point)

Consider the curve $\mathbb{P}^1_{\mathbb{Q}}$, it has function field $\mathbb{Q}(t)$. Let's take the quaternion algebra $A = (t+3, t-1)_{\mathbb{Q}(t)}$.

- We can evaluate A at the point $t = 0$, which gives the quaternion algebra $(3, -1)_{\mathbb{Q}}$.
- Directly trying to evaluate A at $t = 1$ doesn't work since $t - 1 = 0$ there.

  However, we can write A as $A \cong (t+3, -1)_{\mathbb{Q}(t)}$, so we can in face evaluate A at $t = 1$, and get $(4, -1)_{\mathbb{Q}}$.

Examples (Evaluating at a point)

Consider the curve $\mathbb{P}^1_{\mathbb{Q}}$, it has function field $\mathbb{Q}(t)$. Let's take the quaternion algebra $A = (t+3, t-1)_{\mathbb{Q}(t)}$.

- We can evaluate A at the point $t = 0$, which gives the quaternion algebra $(3, -1)_{\mathbb{Q}}$.
- Directly trying to evaluate A at $t = 1$ doesn't work since $t - 1 = 0$ there.
  However, we can write A as $A \cong (t+3, -1)_{\mathbb{Q}(t)}$, so we can in face evaluate A at $t = 1$, and get $(4, -1)_{\mathbb{Q}}$.
- For the point $t = -3$, neither of the above forms of A can be evaluated. Is there some other form for A that we can evaluate?

The residue map

The residue map

Let Y be a codimension-1 (scheme-theoretic) point of X.

The residue map

Let Y be a codimension-1 (scheme-theoretic) point of X. One can then define a *residue map*

$$\partial_Y : \mathrm{Br}(k(X)) \to H^1(k(Y), \mathbb{Q}/\mathbb{Z}).$$

The residue map

Let Y be a codimension-1 (scheme-theoretic) point of X. One can then define a *residue map*

$$\partial_Y : \mathrm{Br}(k(X)) \to H^1(k(Y), \mathbb{Q}/\mathbb{Z}).$$

The kernel of ∂_Y is the set of all elements of $\mathrm{Br}(k(X))$ that can be evaluated at Y.

The residue map

Let Y be a codimension-1 (scheme-theoretic) point of X. One can then define a *residue map*

$$\partial_Y : \mathrm{Br}(k(X)) \to H^1(k(Y), \mathbb{Q}/\mathbb{Z}).$$

The kernel of ∂_Y is the set of all elements of $\mathrm{Br}(k(X))$ that can be evaluated at Y.

Examples

For the $A = (t+3, t-1)$ in the previous example, if we evaluate the residue of A at $t = -3$, we get

$$\partial_{t=-3}(A) = -1$$

which is not the identity. So A cannot be evaluated at $t = -3$.

Brauer group of an elliptic curve

Brauer group of an elliptic curve

Let E be an elliptic curve defined over k, say by equation $y^2 = f(x)$. One can show that we have an isomorphism

Brauer group of an elliptic curve

Let E be an elliptic curve defined over k, say by equation $y^2 = f(x)$. One can show that we have an isomorphism

$$\mathrm{Br}(E) \cong \mathrm{Br}(k) \oplus H^1(k, E).$$

Brauer group of an elliptic curve

Let E be an elliptic curve defined over k, say by equation $y^2 = f(x)$. One can show that we have an isomorphism

$$\mathrm{Br}(E) \cong \mathrm{Br}(k) \oplus H^1(k, E).$$

If we restrict our attention to 2-torsion, we get

$$\mathrm{Br}(E)[2] \cong \mathrm{Br}(k)[2] \oplus H^1(k, E)[2].$$

Brauer group of an elliptic curve

Brauer group of an elliptic curve

The second group can be described using the following:

Brauer group of an elliptic curve

The second group can be described using the following:

- There is a short exact sequence

$$0 \to E(k)/2E(k) \to H^1(k, E[2]) \to H^1(k, E)[2] \to 0.$$

Brauer group of an elliptic curve

The second group can be described using the following:

- There is a short exact sequence

$$0 \to E(k)/2E(k) \to H^1(k, E[2]) \to H^1(k, E)[2] \to 0.$$

- There is an isomorphism

$$H^1(k, E[2]) \cong \ker(\mathrm{Nm} : L^\times/(L^\times)^2 \to k^\times/(k^\times)^2)$$

where $L = k[\theta]/(f(\theta))$.

Brauer group of an elliptic curve

The second group can be described using the following:

- There is a short exact sequence

$$0 \to E(k)/2E(k) \to H^1(k, E[2]) \to H^1(k, E)[2] \to 0.$$

- There is an isomorphism

$$H^1(k, E[2]) \cong \ker(\mathrm{Nm} : L^\times/(L^\times)^2 \to k^\times/(k^\times)^2)$$

where $L = k[\theta]/(f(\theta))$.

So if we have an element of $L^\times$ whose norm is a square, then it gives an element of $\mathrm{Br}(E)[2]$.

Brauer group of an elliptic curve

Setup: E *has equation* $y^2 = f(x)$*, and* $L = K[\theta]/(f(\theta))$*.*

Brauer group of an elliptic curve

Setup: E *has equation* $y^2 = f(x)$*, and* $L = K[\theta]/(f(\theta))$.
Obtaining an element of $\mathrm{Br}(E)[2]$ from an element of $L^\times$ can be quite tricky.

Brauer group of an elliptic curve

Setup: E has equation $y^2 = f(x)$, and $L = K[\theta]/(f(\theta))$.
Obtaining an element of $\mathrm{Br}(E)[2]$ from an element of $L^\times$ can be quite tricky.
One special case is easier:

One special case

The element of $\mathrm{Br}(E)$ corresponding to $a(\theta - b) \in L^\times$, where $a, b \in K$, is represented by the quaternion algebra

$$(f(b), x - b)_{k(E)}.$$

Section 3

Brauer-Manin obstruction

Let X be a projective variety defined over $\mathbb{Q}$. We have

Hasse principle and weak approximation

Let X be a projective variety defined over $\mathbb{Q}$. We have

$$X(\mathbb{Q}) \hookrightarrow \prod_v X(\mathbb{Q}_v) = X(\mathbb{A}_\mathbb{Q}) \text{ (set of adelic points).}$$

Hasse principle and weak approximation

Let X be a projective variety defined over $\mathbb{Q}$. We have

$$X(\mathbb{Q}) \hookrightarrow \prod_v X(\mathbb{Q}_v) = X(\mathbb{A}_\mathbb{Q}) \text{ (set of adelic points).}$$

The *Hasse principle* is the statement

$$X(\mathbb{A}_\mathbb{Q}) \neq \varnothing \quad \Longrightarrow \quad X(\mathbb{Q}) \neq \varnothing.$$

Related is the stronger notion of *weak approximation*, which states that $X(\mathbb{Q})$ is dense in $X(\mathbb{A}_\mathbb{Q})$.

Brauer-Manin obstruction

Brauer-Manin obstruction

Let A be an element of $\mathrm{Br}(X)$.

Let A be an element of $\mathrm{Br}(X)$.
Given an adelic point $P = (P_v)$, for each v we can evaluate A at $P_v \in X(\mathbb{Q}_v)$ to get an element of $\mathrm{Br}(\mathbb{Q}_v)$

Brauer-Manin obstruction

Let A be an element of $\mathrm{Br}(X)$.
Given an adelic point $P = (P_v)$, for each v we can evaluate A at $P_v \in X(\mathbb{Q}_v)$ to get an element of $\mathrm{Br}(\mathbb{Q}_v) \xrightarrow{\text{inv}} \mathbb{Q}/\mathbb{Z}$.

Brauer-Manin obstruction

Let A be an element of $\text{Br}(X)$.
Given an adelic point $P = (P_v)$, for each v we can evaluate A at $P_v \in X(\mathbb{Q}_v)$ to get an element of $\text{Br}(\mathbb{Q}_v) \xrightarrow{\text{inv}} \mathbb{Q}/\mathbb{Z}$.
If P is actually a rational point, then $A(P) \in \text{Br}(\mathbb{Q})$, so we must have

Brauer-Manin obstruction

Let A be an element of $\mathrm{Br}(X)$.
Given an adelic point $P = (P_v)$, for each v we can evaluate A at $P_v \in X(\mathbb{Q}_v)$ to get an element of $\mathrm{Br}(\mathbb{Q}_v) \xrightarrow{\text{inv}} \mathbb{Q}/\mathbb{Z}$.
If P is actually a rational point, then $A(P) \in \mathrm{Br}(\mathbb{Q})$, so we must have

$$\sum_v \mathrm{inv}_v\, A(P_v) = 0.$$

Brauer-Manin obstruction

- If for all adelic points $P \in X(\mathbb{A}_{\mathbb{Q}})$ we have

$$\sum_v \operatorname{inv}_v A(P_v) \neq 0$$

then there cannot be any rational points. So the Hasse principle fails.

Brauer-Manin obstruction

- If for all adelic points $P \in X(\mathbb{A}_{\mathbb{Q}})$ we have

$$\sum_v \operatorname{inv}_v A(P_v) \neq 0$$

then there cannot be any rational points. So the Hasse principle fails.

- If there is some open subset $U \subseteq X(\mathbb{A}_{\mathbb{Q}})$ such that for all $P \in U$ we have

$$\sum_v \operatorname{inv}_v A(P_v) \neq 0$$

then there cannot be any rational points in U. So weak approximation fails.

Example of a Brauer-Manin obstruction

Let E be the elliptic curve given by equation $y^2 = x^3 + 4x^2 - 1$.
Then $L = \mathbb{Q}[\theta]/(\theta^3 + 4\theta^2 - 1)$.

Example of a Brauer-Manin obstruction

Let E be the elliptic curve given by equation $y^2 = x^3 + 4x^2 - 1$.
Then $L = \mathbb{Q}[\theta]/(\theta^3 + 4\theta^2 - 1)$.

- Take the element $\theta \in L^\times$. It has norm 1, which is a square, so it corresponds to an element of $\mathrm{Br}(E)$.

Example of a Brauer-Manin obstruction

Let E be the elliptic curve given by equation $y^2 = x^3 + 4x^2 - 1$.
Then $L = \mathbb{Q}[\theta]/(\theta^3 + 4\theta^2 - 1)$.

- Take the element $\theta \in L^\times$. It has norm 1, which is a square, so it corresponds to an element of $\mathrm{Br}(E)$.
- Using the formula from earlier, we get the quaternion algebra $A = (f(0), x - 0) = (-1, x)_{\mathbb{Q}(E)}$.

Example of a Brauer-Manin obstruction

Let E be the elliptic curve given by equation $y^2 = x^3 + 4x^2 - 1$.
Then $L = \mathbb{Q}[\theta]/(\theta^3 + 4\theta^2 - 1)$.

- Take the element $\theta \in L^\times$. It has norm 1, which is a square, so it corresponds to an element of $\mathrm{Br}(E)$.
- Using the formula from earlier, we get the quaternion algebra $A = (f(0), x - 0) = (-1, x)_{\mathbb{Q}(E)}$.
- For all non-archimedean places v, we find that $\mathrm{inv}_v A(P_v) = 0$ for all $P_v \in E(\mathbb{Q}_v)$.

Example of a Brauer-Manin obstruction

Let E be the elliptic curve given by equation $y^2 = x^3 + 4x^2 - 1$.
Then $L = \mathbb{Q}[\theta]/(\theta^3 + 4\theta^2 - 1)$.

- Take the element $\theta \in L^\times$. It has norm 1, which is a square, so it corresponds to an element of $\mathrm{Br}(E)$.
- Using the formula from earlier, we get the quaternion algebra $A = (f(0), x - 0) = (-1, x)_{\mathbb{Q}(E)}$.
- For all non-archimedean places v, we find that $\operatorname{inv}_v A(P_v) = 0$ for all $P_v \in E(\mathbb{Q}_v)$.
- For v the archimedean place, we find

$$\operatorname{inv}_v A(P_v) = \begin{cases} 0 & x > 0 \\ \frac{1}{2} & x < 0 \end{cases}.$$

Example of a Brauer-Manin obstruction

Let E be the elliptic curve given by equation $y^2 = x^3 + 4x^2 - 1$.
Then $L = \mathbb{Q}[\theta]/(\theta^3 + 4\theta^2 - 1)$.

- Take the element $\theta \in L^\times$. It has norm 1, which is a square, so it corresponds to an element of $\mathrm{Br}(E)$.
- Using the formula from earlier, we get the quaternion algebra $A = (f(0), x - 0) = (-1, x)_{\mathbb{Q}(E)}$.
- For all non-archimedean places v, we find that $\mathrm{inv}_v A(P_v) = 0$ for all $P_v \in E(\mathbb{Q}_v)$.
- For v the archimedean place, we find

$$\mathrm{inv}_v A(P_v) = \begin{cases} 0 & x > 0 \\ \frac{1}{2} & x < 0 \end{cases}.$$

- Therefore there are no rational points with $x < 0$.

Section 4

Elliptic surfaces

Elliptic surfaces

Now suppose we have an equation $y^2 = x^3 + ax^2 + bx + c$ but with $a, b, c \in \mathbb{Q}(t)$. This can be viewed in two ways:

Elliptic surfaces

Now suppose we have an equation $y^2 = x^3 + ax^2 + bx + c$ but with $a, b, c \in \mathbb{Q}(t)$. This can be viewed in two ways:

- We can view this as describing an elliptic curve E defined over $\mathbb{Q}(t)$.

Elliptic surfaces

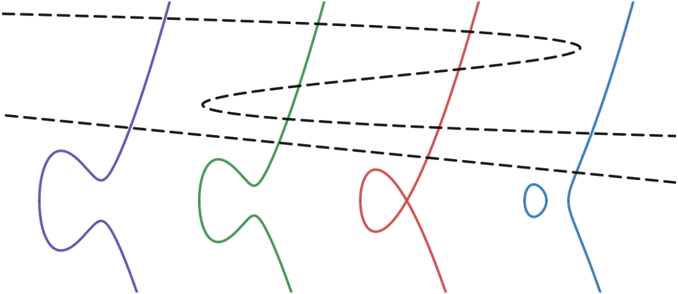
Now suppose we have an equation $y^2 = x^3 + ax^2 + bx + c$ but with $a, b, c \in \mathbb{Q}(t)$. This can be viewed in two ways:

- We can view this as describing an elliptic curve E defined over $\mathbb{Q}(t)$.
- We can view this as an equation in 3 variables, describing a surface X.

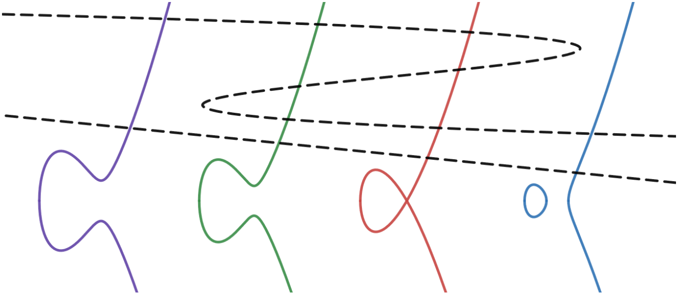
Elliptic surfaces

Now suppose we have an equation $y^2 = x^3 + ax^2 + bx + c$ but with $a, b, c \in \mathbb{Q}(t)$. This can be viewed in two ways:

- We can view this as describing an elliptic curve E defined over $\mathbb{Q}(t)$.
- We can view this as an equation in 3 variables, describing a surface X.

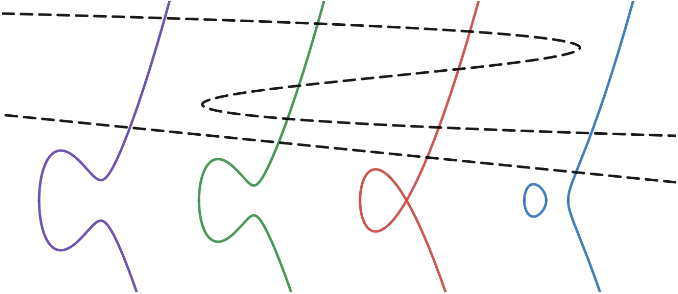


Elliptic surfaces



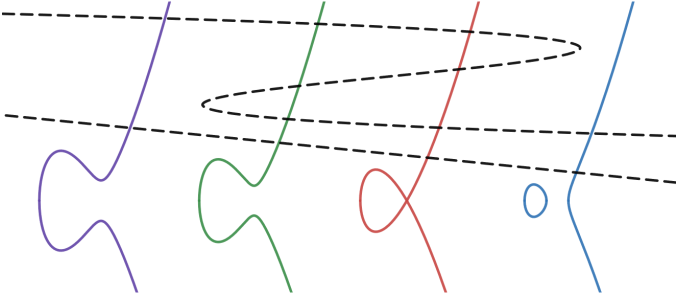
Most of the fibres will be elliptic curves (good fibres), so this is an elliptic fibration.

Elliptic surfaces



Most of the fibres will be elliptic curves (good fibres), so this is an elliptic fibration.
Some of the fibres will be singular (bad fibres).

Elliptic surfaces



Most of the fibres will be elliptic curves (good fibres), so this is an elliptic fibration.

Some of the fibres will be singular (bad fibres).

We resolve all singularities of X by blowing them up to get a smooth surface $\mathcal{E}$..

Brauer elements of elliptic surfaces

Brauer elements of elliptic surfaces

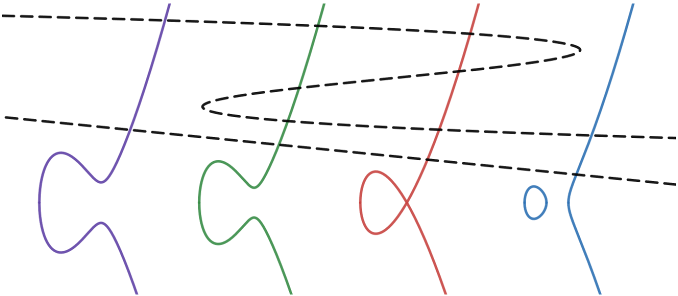
We want to find elements of $\mathrm{Br}(\mathcal{E})$ to obtain examples of Brauer-Manin obstructions.
We do this by finding all algebras $A \in \mathrm{Br}(k(\mathcal{E}))$ that have zero residue on every irreducible curve on $\mathcal{E}$.

Brauer elements of elliptic surfaces

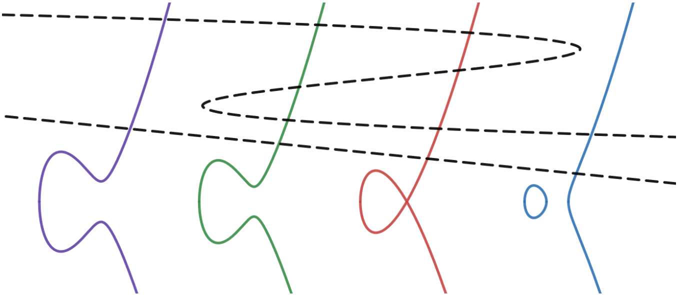
We want to find elements of $\mathrm{Br}(\mathcal{E})$ to obtain examples of Brauer-Manin obstructions.
We do this by finding all algebras $A \in \mathrm{Br}(k(\mathcal{E}))$ that have zero residue on every irreducible curve on $\mathcal{E}$.
We have an inclusion

$$\mathrm{Br}(\mathcal{E}) \subseteq \mathrm{Br}(E) \subseteq \mathrm{Br}(k(\mathcal{E})).$$

Brauer elements of elliptic surfaces



Brauer elements of elliptic surfaces



The points of E correspond to the horizontal curves in $\mathcal{E}$, so if $A \in \mathrm{Br}(E)$ then we only need to check the vertical curves (i.e. components of fibres).

Brauer elements of elliptic surfaces

Brauer elements of elliptic surfaces

In our situation, the base field of the elliptic curve E is $\mathbb{Q}(t)$, the function field of the projective line.

Brauer elements of elliptic surfaces

In our situation, the base field of the elliptic curve E is $\mathbb{Q}(t)$, the function field of the projective line.
$L = \mathbb{Q}(t)[\theta]/(f(\theta))$ is the function field of the curve C

$$C : x^3 + ax^2 + bx + c = 0$$

with a degree 3 map to $\mathbb{P}^1$.

Brauer elements of elliptic surfaces

In our situation, the base field of the elliptic curve E is $\mathbb{Q}(t)$, the function field of the projective line.
$L = \mathbb{Q}(t)[\theta]/(f(\theta))$ is the function field of the curve C

$$C : x^3 + ax^2 + bx + c = 0$$

with a degree 3 map to $\mathbb{P}^1$.
Let $\alpha \in L^\times$ be an element with square norm, and A the corresponding element of $\mathrm{Br}(E)$.

Brauer elements of elliptic surfaces

In our situation, the base field of the elliptic curve E is $\mathbb{Q}(t)$, the function field of the projective line.
$L = \mathbb{Q}(t)[\theta]/(f(\theta))$ is the function field of the curve C

$$C : x^3 + ax^2 + bx + c = 0$$

with a degree 3 map to $\mathbb{P}^1$.
Let $\alpha \in L^\times$ be an element with square norm, and A the corresponding element of $\mathrm{Br}(E)$.

Residue map condition

On points $t_0 \in \mathbb{P}^1$ where $\mathcal{E}$ has good reduction, we have

$$\partial_{t=t_0}(A) = 1 \iff \mathrm{ord}_P(\alpha) \text{ is even for all } P \text{ above } t_0.$$

Our process for finding elements of $\mathrm{Br}(\mathcal{E})[2]$:

Our process for finding elements of $\mathrm{Br}(\mathcal{E})[2]$:

- Start with the elliptic curve E defined over $\mathbb{Q}(t)$.

Brauer elements of elliptic surfaces

Our process for finding elements of $\mathrm{Br}(\mathcal{E})[2]$:

- Start with the elliptic curve E defined over $\mathbb{Q}(t)$.
- An element $\alpha \in L^{\times}$ corresponds to an element A of $\mathrm{Br}(E)[2]$.

Brauer elements of elliptic surfaces

Our process for finding elements of $\mathrm{Br}(\mathcal{E})[2]$:

- Start with the elliptic curve E defined over $\mathbb{Q}(t)$.
- An element $\alpha \in L^{\times}$ corresponds to an element A of $\mathrm{Br}(E)[2]$.
- The condition that A must have zero residue on good fibres corresponds to a condition on the order of vanishing of α at points.

Brauer elements of elliptic surfaces

Our process for finding elements of $\mathrm{Br}(\mathcal{E})[2]$:

- Start with the elliptic curve E defined over $\mathbb{Q}(t)$.
- An element $\alpha \in L^\times$ corresponds to an element A of $\mathrm{Br}(E)[2]$.
- The condition that A must have zero residue on good fibres corresponds to a condition on the order of vanishing of α at points.
- Imposing these conditions reduces us to a finite subgroup of $L^\times/(L^\times)^2$.

Brauer elements of elliptic surfaces

Our process for finding elements of $\mathrm{Br}(\mathcal{E})[2]$:

- Start with the elliptic curve E defined over $\mathbb{Q}(t)$.
- An element $\alpha \in L^\times$ corresponds to an element A of $\mathrm{Br}(E)[2]$.
- The condition that A must have zero residue on good fibres corresponds to a condition on the order of vanishing of α at points.
- Imposing these conditions reduces us to a finite subgroup of $L^\times/(L^\times)^2$.
- Check the conditions on each component of the bad fibres to determine which elements lie in $\mathrm{Br}(\mathcal{E})$.

Brauer-Manin obstruction on an elliptic surface

With this method, we were able to find an example of a Brauer-Manin obstruction to weak approximation on an elliptic surface over $\mathbb{Q}$.

Brauer-Manin obstruction on an elliptic surface

With this method, we were able to find an example of a Brauer-Manin obstruction to weak approximation on an elliptic surface over $\mathbb{Q}$.
Take the surface $\mathcal{E}$ and the algebra $A \in \mathrm{Br}(\mathcal{E})$

$$\mathcal{E} : y^2 + (1 + t - t^2)xy + (t^2 - t^3)y = x^3 + (t^2 - t^3)x^2$$

$$A = \left(t^2 - t + 1,\ x - \frac{t^3 - t^2}{t^2 - t + 1}\right).$$

Brauer-Manin obstruction on an elliptic surface

With this method, we were able to find an example of a Brauer-Manin obstruction to weak approximation on an elliptic surface over $\mathbb{Q}$.
Take the surface $\mathcal{E}$ and the algebra $A \in \mathrm{Br}(\mathcal{E})$

$$\mathcal{E} : y^2 + (1 + t - t^2)xy + (t^2 - t^3)y = x^3 + (t^2 - t^3)x^2$$

$$A = \left(t^2 - t + 1,\ x - \frac{t^3 - t^2}{t^2 - t + 1} \right).$$

Then $\sum_v \mathrm{inv}_v A(P) = \frac{1}{2}$, where P is the adelic point given by

$$P_v = \begin{cases} (2, -1 + \sqrt{57}, -2) & v = 2 \\ (2, 4, 2) & v \neq 2. \end{cases}$$

Brauer-Manin obstruction on an elliptic surface

With this method, we were able to find an example of a Brauer-Manin obstruction to weak approximation on an elliptic surface over $\mathbb{Q}$.
Take the surface $\mathcal{E}$ and the algebra $A \in \mathrm{Br}(\mathcal{E})$

$$\mathcal{E} : y^2 + (1 + t - t^2)xy + (t^2 - t^3)y = x^3 + (t^2 - t^3)x^2$$

$$A = \left(t^2 - t + 1,\ x - \frac{t^3 - t^2}{t^2 - t + 1} \right).$$

Then $\sum_v \mathrm{inv}_v A(P) = \frac{1}{2}$, where P is the adelic point given by

$$P_v = \begin{cases} (2, -1 + \sqrt{57}, -2) & v = 2 \\ (2, 4, 2) & v \neq 2. \end{cases}$$

So there are no rational points in a neighbourhood of P.

Next steps

- Use similar methods to find examples of failure of the Hasse principle.
- Look at higher torsion, e.g. $\mathrm{Br}(E)[3], \mathrm{Br}(E)[4]$.

Thanks for listening!