Descent on elliptic surfaces and the Brauer-Manin obstruction

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So if there are no adelic points, then there cannot be any rational points.

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The Hasse principle and weak approximation

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Related is the stronger notion of *weak approximation*, which states that $X(k)$ is dense in $X(\mathbb{A}_k)$.
These are true for some classes of varieties (e.g. conics), but not all varieties. It's an interesting question to find examples where these do not hold.

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An example of a central simple algebra is a *cyclic algebra*. If k contains ζ a primitive n^{th} root of unity, then given $a, b \in k^\times$, the algebra generated by i, j with relations

$$i^n = a, \quad j^n = b, \quad ij = \zeta ji$$

is a cyclic algebra, denoted $(a, b)_n$.

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- $\mathrm{Br}(k_v) \cong \mathbb{Q}/\mathbb{Z}$ for k_v a local field.
  There is a canonical isomorphism $\mathrm{inv}_v : \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$, called the *invariant map*.

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  There is a canonical isomorphism $\mathrm{inv}_v : \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$, called the *invariant map*.
- For k a number field, we have

Theorem (Albert-Brauer-Hasse-Noether)

The sequence

$$0 \to \mathrm{Br}(k) \to \bigoplus_v \mathrm{Br}(k_v) \xrightarrow{\sum \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0$$

is exact.

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This can be done for adelic points too: there is a pairing

$$X(\mathbb{A}_k) \times \mathrm{Br}(X) \to \bigoplus_v \mathrm{Br}(k_v).$$

Brauer-Manin obstruction

By summing the local invariants, one obtains a pairing

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Since the sum of local invariants of an element of $\mathrm{Br}(k)$ is zero, we get that the above pairing is zero for all points of $X(k) \subseteq X(\mathbb{A}_k)$. This gives a new way to prove the nonexistence of rational points: if we find an element $A \in \mathrm{Br}(X)$ such that $A(P)$ is nonzero for all adelic points $P \in X(\mathbb{A}_k)$, then none of the adelic points can be rational points. We call this a *Brauer-Manin obstruction* to the Hasse principle.

Brauer-Manin obstruction

More generally, we can define the *Brauer-Manin set*

$$X(\mathbb{A}_k)^{\mathrm{Br}} = \{P \in X(\mathbb{A}_k) : \sum \mathrm{inv}\, A(P) = 0 \text{ for all } A \in \mathrm{Br}(X)\}.$$

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We then have the inclusions

$$X(k) \subseteq \overline{X(k)} \subseteq X(\mathbb{A}_k)^{\mathrm{Br}} \subseteq X(\mathbb{A}_k).$$

- If $X(\mathbb{A}_k)$ is nonempty but $X(\mathbb{A}_k)^{\mathrm{Br}}$ is empty, then there is a *Brauer-Manin obstruction to the Hasse principle*.
- If $X(\mathbb{A}_k)^{\mathrm{Br}} \neq X(\mathbb{A}_k)$, then there is a *Brauer-Manin obstruction to weak approximation*.

Example

Consider the elliptic curve $E : y^2 = x^3 + 4x^2 - 1$.

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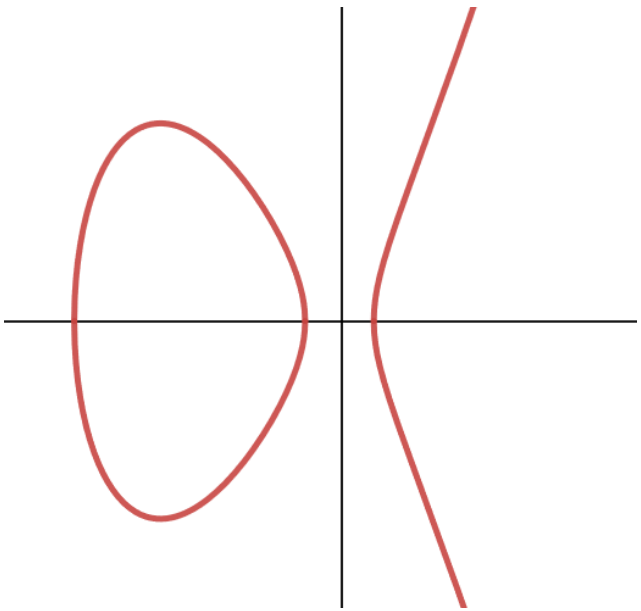

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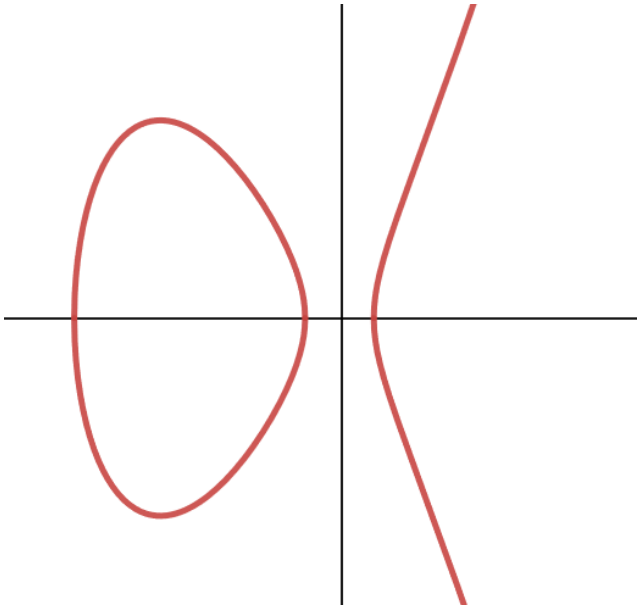

Take the element of $\mathrm{Br}(E)$ represented by the cyclic algebra $A = (-1, x)_2$. Then for all adelic points P with $(v = \infty)$-part lying on the left component, we have $\sum \mathrm{inv}\, A(P) = \frac{1}{2}$.

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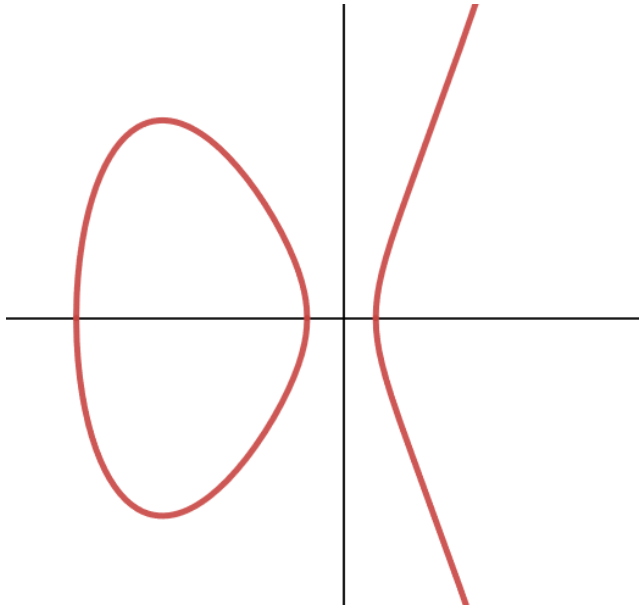

Take the element of $\mathrm{Br}(E)$ represented by the cyclic algebra $A = (-1, x)_2$. Then for all adelic points P with $(v = \infty)$-part lying on the left component, we have $\sum \mathrm{inv}\, A(P) = \frac{1}{2}$.
This shows that there are no rational points on the left component, so there is a Brauer-Manin obstruction to weak approximation.

Summary

- The *Brauer-Manin obstruction to weak approximation* is when

$$X(\mathbb{A}_k)^{\mathrm{Br}} \subsetneq X(\mathbb{A}_k).$$

- We wish to find elements of the Brauer group of a scheme X to find examples of *Brauer-Manin obstruction*.

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$$X(\mathbb{A}_k)^{\mathrm{Br}} \subsetneq X(\mathbb{A}_k).$$

- We wish to find elements of the Brauer group of a scheme X to find examples of *Brauer-Manin obstruction*.
- Some elements of the Brauer group of a field can be represented by *cyclic algebras* $(a, b)_n$.
- If we can find an adelic point P and a Brauer element A such that $\sum \mathrm{inv}_v A(P) \neq 0$, then we have a Brauer-Manin obstruction to weak approximation.

Calculating the Brauer group of a scheme

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Residue map

Let $D \subseteq X$ be an irreducible codimension-1 subvariety. Then there is a *residue map*

$$\partial_D : \mathrm{Br}(k(X)) \to H^1(k(D), \mathbb{Q}/\mathbb{Z}).$$

The kernel of this map is the subgroup of elements that can be evaluated at D.

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Let $A \in \mathrm{Br}(k(X))$. Then $A \in \mathrm{Br}(X)$ if and only if $\partial_D(A) = 0$ for all irreducible codimension-1 subvarieties $D \subseteq X$.

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This theorem allows us to find elements of $\mathrm{Br}(X)$ by just checking all the residue maps.

Brauer group of an elliptic curve

For an elliptic curve E over a field k, there is an isomorphism

$$\frac{\mathrm{Br}(E)}{\mathrm{Br}_0(E)} \cong H^1(k, E).$$

Here $\mathrm{Br}_0(E)$ are the *constant algebras*, the image of the natural map $\mathrm{Br}(k) \to \mathrm{Br}(E)$.

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The group $H^1(k, E)$ is well understood, so we can use this to find elements easily. However, calculating the above isomorphism explicitly is not entirely straightforward.

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For example, take C to be $\mathbb{P}^1$, parametrised by z. Then we can write down a Weierstrass equation with coordinates in $k(z)$, such as

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This defines an elliptic surface $\mathcal{E}$. The special fibre above a point $z = z_0$ is given by substituting in z_0 into the above equation.

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For elements of $\mathrm{Br}(E)$, the residues with respect to horizontal curves will automatically be zero, so one only needs to check the residues with respect to vertical curves.

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4. If all the residues are trivial, we have found an element of $\mathrm{Br}(\mathcal{E})$
5. Hope the element we found obstructs weak approximation

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Let's restrict ourselves to looking at 3-torsion. Then we have the exact sequence from Galois cohomology

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The group $H^1(K, E[3])$ is easier to describe.

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Let S be the subvariety of $\mathcal{E}$ consisting of all 3-torsion points of all the fibres. This will be a union of curves.

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Then $H^1(K, E[3])$ is isomorphic to a subgroup of $R^\times$ modulo cubes.

3-descent on an elliptic curve - Example 1

Let $T_1, T_2 \in E$ be points generating $E[3]$.
If T_1, T_2 are both defined over K, then S consists of 9 sections, so $R = K^9$.

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If T_1, T_2 are both defined over K, then S consists of 9 sections, so $R = K^9$. In this case, elements of $H^1(K, E[3]) \subseteq R^\times$ are determined by the values on the two components S_1, S_2 corresponding to the points T_1, T_2, so we may write elements of $H^1(K, E[3])$ as pairs (a_1, a_2).

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The corresponding Brauer element of (a_1, a_2) is given by the sum of cyclic algebras

$$(a_1, f_2)_3 + (a_2, f_1)_3$$

where f_i is a function on E with divisor $3(T_i) - 3(O)$.

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Suppose T_1 is defined over K but T_2 is not. Then S decomposes into 5 sections: S_0, S_1, S_2 corresponding to $O, T_1, 2T_1$, and S_3, S_4 corresponding to the Galois orbits $\{T_2 + nT_1\}, \{2T_2 + nT_1\}$.

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The function fields of S_0, S_1, S_2 are all isomorphic to K, while the function fields of S_3, S_4 are isomorphic to L a degree 3 Galois extension of K.
In this case, elements of $H^1(K, E[3]) \subseteq R^\times$ are determined by the values on the two components S_1, S_3, so we may write elements of $H^1(K, E[3])$ as pairs (a, b) where $a \in K, b \in L$.

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$$\left(af_1, \mathrm{Tr}_{L/K}(bf_2) - 3\frac{bd}{\sigma^2(d)}\frac{f_2 d'}{\sigma^2(d')}\right)_3$$
$$-\left(a, \mathrm{Tr}_{L/K}(b) - 3\frac{bd}{\sigma^2(d)}\right)_3 - \left(f_1, \mathrm{Tr}_{L/K}(f_2) - 3\frac{f_2 d'}{\sigma^2(d')}\right)_3$$

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where

- σ is a generator of $\mathrm{Gal}(L/K)$
- f_i is a function on E with divisor $3(T_i) - 3(O)$
- d is a cube root of $a\sigma(b)/b$
- d' is a cube root of $f_1\sigma(f_2)/f_2$

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This can then be translated into a local condition on $a \in H^1(K, E[3])$.

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- If $\mathcal{E}$ has multiplicative reduction at P, then let Q_0, Q_1, Q_2 be the intersections of S with the identity component of the special fibre above P. Then the resulting local condition is
  $$\mathrm{ord}_{Q_i}(a) \equiv 0 \mod 3, \quad a(Q_i) \text{ is a perfect cube.}$$

The Selmer group

Imposing all these local conditions will give us a subgroup of $H^1(K, E[3])$ that maps surjectively onto $(\mathrm{Br}(\mathcal{E})/\mathrm{Br}_0(\mathcal{E}))[3]$.

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This is analagous to the Selmer group from performing descent on an elliptic curve defined over a number field. The Brauer group then plays the part of the Tate-Shafarevich group.

Example

Let $k = \mathbb{Q}(\zeta_3)$, and let $\mathcal{E}$ be the elliptic surface given by equation

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- The bad fibres have reduction types

0	1	-1	$\pm i$	∞
I_{12}	I_1	I_1	I_1	IV^*

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Setting $a = \frac{t-1}{t+1}$ satisfies all of these local conditions.

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Therefore A must take a nonzero value in a neigbourhood of the adelic point given by

$$P = \begin{cases} \text{identity point on a fibre} & v \neq 3 \\ (t = 1 - \zeta_3, x = 1, y = \ldots) & v = 3 \end{cases}$$

So A gives a Brauer-Manin obstruction to weak approximation on $\mathcal{E}$.

Thanks for listening!